The energy-momentum of a quantum field in a cavity does not transform as a Lorentz 4-vector

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Abstract

It is shown that the field energy and momentum of a massless quantized system confined to a cavity do not transform as a Lorentz 4-vector. This is a consequence of the fact that there is no Lorentz-invariant way of separating the cavity-internal stress tensor from the confining stress tensor. An analysis is done of blackbody radiation and of the Casimir effect, both for a 1-D massless field and for 3-D electromagnetism.

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Introduction

In this note, we consider quantum fields confined to a cavity with perfectly reflecting interior and exterior walls. The finite-temperature field energy is easily computed, including zero-point effects leading to the Casimir effect. The internal energy and momentum are shown to be *non-vectorial*, by which we mean that the four quantities do **not** transform as components of a Lorentz 4-vector. A similar observation has previously been made by McDonald [1] and Celmaster [2] for classical waves in a cavity. This non-vectorial behavior of internal energy and momentum, is a general property of classical confinement. It is easiest to visualize the origin of the non-vector terms, by considering particles bouncing inside a box. If, in one reference frame, two particles happen to simultaneously collide with opposite walls, then in another reference frame those collisions won't be simultaneous. In fact, if we count the number of particles simultaneously traveling in opposite directions, it turns out that this number is frame-dependent. Because of that, there isn't a Lorentz invariant way to separate the internal energy and momentum of the particles from the energy and momentum of the walls. The total energy and momentum (walls plus internal subsystem) form a Lorentz 4-vector, but not the internal components.

1 Review of classical confinement

The key results of McDonald [1] and Celmaster [2] pertain to classical electromagnetic standing waves in a right parallelepiped cavity with perfectly reflecting walls. The cavity coordinates can be taken as $(0 < x_1 < L_1, 0 < x_2 < L_2, 0 < x_3 < L_3)$. Modes are indexed by $\mathbf{n} = (n_1, n_2, n_3)$. Set the mode vector[1] components and frequencies to $k_i^{\mathbf{n}} = n_i \pi / L_i$ and $\omega^{\mathbf{n}} = \sqrt{\sum_{i=1}^{3} (k_i^{\mathbf{n}})^2}$, where the n_i are non-negative integers (of which at least 2 must be nonzero).

Then the rest-frame standing waves (modes) for the i^{th} component of the electric and magnetic fields are

$$E_i(t, \mathbf{x}) = \sum_{\mathbf{n}} \left[\frac{1}{\tan(k_i^{\mathbf{n}} x_i)} \prod_{j=1}^{3} \sin(k_j^{\mathbf{n}} x_j) \right] [a_i^{\mathbf{n}} \sin(\omega^{\mathbf{n}} t) + b_i^{\mathbf{n}} \cos(\omega^{\mathbf{n}} t)], \tag{1}$$

$$B_i(t, \mathbf{x}) = \sum_{\mathbf{n}} \left[\frac{\tan(k_i^{\mathbf{n}} x_i)}{\omega^{\mathbf{n}}} \prod_{j=1}^{3} \cos(k_j^{\mathbf{n}} x_j) \right] [-(\mathbf{a}^{\mathbf{n}} \times \mathbf{k}^{\mathbf{n}})_i \cos(\omega^{\mathbf{n}} t) + (\mathbf{b}^{\mathbf{n}} \times \mathbf{k}^{\mathbf{n}})_i \sin(\omega^{\mathbf{n}} t)], \tag{2}$$

where $\mathbf{a}^{\mathbf{n}}$ and $\mathbf{b}^{\mathbf{n}}$ are coefficient vectors each perpendicular to $\mathbf{n}$.[2] The energy is computed in the rest-frame as

$$U(t) = \frac{1}{2} \int_0^{L_1} \int_0^{L_2} \int_0^{L_3} (|\mathbf{E}(t, \mathbf{x})|^2 + |\mathbf{B}(t, \mathbf{x})|^2) d^3 x. \tag{3}$$

Similarly, the momentum in the rest-frame can be obtained by computing the Poynting vector which is seen to be 0 since it is a sum of equal but opposite contributions from the left- and right-moving waves that make up the standing wave.

Now go into a reference frame moving at velocity v to the right and compute the energy of the field in this new reference frame, which we will call the $'$ frame.[3] Let $\gamma(v) = 1/\sqrt{1 - v^2}$. The Lorentz transformations of these electromagnetic fields to the $'$-frame, with coordinates $x' = \gamma(v)(x - vt)$ and $t' = \gamma(v)(t - vx)$, are

$$\begin{aligned} E_1'(t', x', y', z') &= E_1(\gamma(v)(t' + vx'), \gamma(v)(x' + vt'), y', z'), \\ B_1'(t', x', y', z') &= B_1(\gamma(v)(t' + vx'), \gamma(v)(x' + vt'), y', z'), \end{aligned} \tag{4}$$

[1] We use the term "mode vector" to denote the vector whose components are magnitudes of the wave vectors for the oppositely moving waves that constitute the standing wave in question.

[2] Notice that although the combination of trigonometric functions appears to sometimes be ill-defined owing to either a 0 or infinite value of the tangent, it turns out that the trigonometric ratios all have perfectly finite limits as the tangent approaches those values.

[3] Quantities in the $'$ frame will be written with a $'$, while quantities in the rest frame of the box will be written without a $'$.

and for $i = 2$ or 3

$$\begin{aligned} E_i'(t', x', y', z') &= \gamma(v)[E_i(\gamma(v)(t' + vx'), \gamma(v)(x' + vt'), y', z') \\ &\quad + \epsilon_{i1j} v B_j(\gamma(v)(t' + vx'), \gamma(v)(x' + vt'), y', z')], \\ B_i'(t', x', y', z') &= \gamma(v)[B_i(\gamma(v)(t' + vx'), \gamma(v)(x' + vt'), y', z') \\ &\quad - \epsilon_{i1j} v E_j(\gamma(v)(t' + vx'), \gamma(v)(x' + vt'), y', z')], \end{aligned} \tag{5}$$

where, in these equations, there is implied summation over j with the standard Levi-Civita epsilon symbol ϵ_{i1j}.

The energy in the $'$-frame is[4]

$$U'(t') = \frac{1}{2} \int_{-vt'}^{\frac{L_1}{\gamma(v)} - vt'} \left[\int_0^{L_2} \int_0^{L_3} (|\mathbf{E}'(t', x', y', z')|^2 + |\mathbf{B}'(t', x', y', z')|^2) dy' dz' \right] dx'. \tag{6}$$

After applying all the transformations, it can be shown that

$$\langle U' \rangle = \gamma(v) \left[1 + u_x^2 v^2 \right] \langle U \rangle , \tag{7}$$

where $u_x = k_1/\omega$ is the x-component of the phase velocity of the right-moving waves of eqs. (1) and (2), and $\langle U \rangle$ denotes the time average of $U(t)$.

If it were possible to treat a standing wave as a standalone subsystem, then, since its rest momentum is 0, we would have expected the boosted energy to be $\gamma(v)$ times the rest energy. However, we see from eq. (7), that $\langle U \rangle$ does not transform that way.

As mentioned in the Introduction, the non-vectorial nature of the internal energy and momentum isn't specific to electromagnetic waves. A similar effect is found for particles in a box. Imagine a box of length L, in which there are particles of mass m and velocities $(\pm u_x, \pm u_y, \pm u_z)$ bouncing elastically off the walls. We will again consider both the box rest-frame, and the $'$-frame where the observer moves in the positive x direction with speed v. In the rest frame, the left-moving particles are uniformly spaced in the x-direction with separations $2L/N$, and the right-moving particles have the same separations. The interval in time between left-moving (or right-moving) particles passing a given value of x is $\Delta = 2L/(u_x N)$ and at any instant (other than during

[4]There is an arbitrariness to the time-dependence of the x' integral. The difference between upper and lower bounds must be equal to the contracted length L_1/γ but the choice of time-displacement depends on precisely which value of t' we choose to correspond to t. In the moving frame, the positions (t', x') for a fixed value of t' correspond in the rest frame to different values of t. Our choice of bounds doesn't have an intuitive interpretation, but is as good as any other. All time-dependent results will be correct up to a constant shift in time.

the times when particles collide with the walls) there are N particles in flight. Assume that $m > 0$. Each particle has a rest frame 4-momentum $m\gamma(u)(1, \pm u_x, \pm u_y, \pm u_z)$ where $u = \sqrt{u_x^2 + u_y^2 + u_z^2}$. In the rest frame, the particle subsystem total energy is then

$$E = Nm\gamma(u) = \frac{2Lm\gamma(u)}{u_x \Delta}. \tag{8}$$

In the $'$-frame the particle subsystem energy becomes

$$E' = N\gamma(v)m\gamma(u)(1 + u_x^2 v^2) = \gamma(v)E(1 + u_x^2 v^2). \tag{9}$$

The non-vectorial term in this equation (proportional to v^2) is the same as in eq. (7).

As mentioned earlier, the Lorentz 4-vector nature of energy and momentum, is restored when considering the entire system consisting of the cavity and radiation (or box and particles). If the walls were free to move, then the radiation pressure (or collisions) would push them apart and give them extra energy. That energy can be computed both in the rest frame and the $'$-frame, and it can be shown (for the case of particles in a box) to have a non-vectorial term which exactly cancels that of the subsystem. It could be argued that this analysis strictly only applies to the situation where the walls are free to move. So an alternate analysis can be given. Instead of moving walls, let the walls be static and kept in place by external illumination. Then the total energy of interest is the sum of the internal and the external energy (the box energy remains constant). Again, it can be shown that the total energy and momentum transform as a 4-vector.

2 Quantum treatment of a one-dimensional massless scalar field theory

On general grounds, one might expect that the quantum field energy would have the same kind of transformation properties as the classical field energy and in particular, that confined fields in the quantum theory, would have a non-vectorial energy term. Still, the calculations are different. In particular, there is a Casimir effect that arises from field contributions that are both interior and exterior to the cavity.

Let's examine a one-dimensional free real scalar field theory in the region $0 \leq x \leq L$ with Dirichlet boundary conditions at $x = 0$ and $x = L$. We will refer to such Dirichlet boundaries as *perfect mirrors*. The field operator is

$\Phi_L(t,x)$ with $\Phi_L(t,0)=0$ and $\Phi_L(t,L)=0$. Field values are set to 0 outside this region. Equations of motion within the region are satisfied by

$$\Phi_L(t,x) = \sum_n i\sqrt{\frac{1}{n\pi}}\sin(\frac{n\pi x}{L})(a_n e^{-in\pi t/L} - a_n^\dagger e^{in\pi t/L}), \tag{10}$$

where the index n ranges over positive integers. As usual, the raising and lowering operators in this equation obey commutation relations

$$[a_n, a_{n'}^\dagger] = \delta_{n,n'}, \tag{11}$$

and the theory is instantiated on a Fock space generated from the vacuum state and from one-particle states of the form

$$|n\rangle = a_n^\dagger|0\rangle. \tag{12}$$

In this scalar theory, the Hamiltonian density[5] is

$$\tilde{H}_S(x,t) = \frac{1}{2}\left((\partial_t\Phi_L(t,x))^2 + (\partial_x\Phi_L(t,x))^2\right). \tag{13}$$

Since the terms on the right are products of operators, the expression is understood to be regularized. There are of course many methods that can be used for regularization. We'll adopt the point-splitting method (see, for example, Birrell and Davies [3] or Polchinski [4] (the operator product expansion)). When applied to eq. (13), the expression for the Hamiltonian density becomes[6]

$$\tilde{H}_S(t,x) = \lim_{\epsilon\to 0}\frac{1}{2}\left[\{\partial_t\Phi_L(t,x+\epsilon)\partial_t\Phi_L(x)\}_S + \{\partial_x\Phi_L(t,x+\epsilon)\partial_x\Phi_L(t,x)\}_S\right], \tag{14}$$

where we use the notation $\{AB\}_S$ to denote the symmetrized product $\frac{1}{2}(AB+BA)$. The total Hamiltonian, is

$$H_S = \int_0^L dx\tilde{H}_S(t,x). \tag{15}$$

[5]Herein, the term *Hamiltonian* refers to the operator for the time-translation generator, and the term *Energy* refers to a context-dependent average value with respect to states.

[6]The point-splitting method as applied here, is to be interpreted so that the operators whose arguments involve ϵ are to be formally treated as though the cavity is displaced by a value of ϵ. This interpretation is at best vague, since we want all terms in an operator-product to act on the same Hilbert Space. More accurately, we simply consider a prescription where certain terms that would have involved the parameter x in expectation values of the Hamiltonian density, are modified to $x+\epsilon$. Were we to integrate the Hamiltonian density over the range $0 < x < X-\epsilon$, we would end up with a change to the finite Casimir energy term. Some other regularization methods appear to have better physical motivations.

Energy is computed by taking expectation values of the operator H_S in various states of interest. Because of this, and also because of the necessity for renormalization, the cavity analysis should be slightly modified from what was done in the classical case. Following the usual approach when calculating the Casimir effect, we will consider a system of length L as above, but within which is a pair of perfect mirrors separated by distance X in the rest frame. Those mirrors are free to move[7] and therefore X may change with time. The position of the left mirror will be given as $x = x_0$ and the position of the right mirror as $x = x_0 + X$. For convenience, take $x_0 = 0$. The Dirichlet boundary conditions are now $\Phi_L^X(t, 0) = 0$, $\Phi_L^X(t, X) = 0$ and $\Phi_L^X(t, L) = 0$.

The field operator $\Phi_L^X(t, x)$ satisfying appropriate commutation relations, equations of motion and boundary conditions can be written as a tensor product of operators on the Hilbert Space $\mathcal{H}$ defined as

$$\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_{L-X}. \tag{16}$$

where $\mathcal{H}_Z$ denotes the Fock space generated by one-particle states of the form given in equation (12). Then

$$\Phi_L^X(t, x) = \Phi_X(t, x) \otimes \Phi_{L-X}(t, x), \tag{17}$$

where $\Phi_Z(t, x)$ operates on the Hilbert space $\mathcal{H}_Z$[8] ,

$$\Phi_X(t, x) = \sum_n i\sqrt{\frac{1}{n\pi}} \sin(\frac{n\pi x}{X})(a_n e^{-in\pi t/X} - a_n^\dagger e^{in\pi t/X}), \tag{18}$$

for $0 \leq x \leq X$, and

$$\Phi_{L-X}(t, x) = \sum_n i\sqrt{\frac{1}{n\pi}} \sin(\frac{n\pi x}{L-X})(a_n e^{-in\pi t/(L-X)} - a_n^\dagger e^{in\pi t/(L-X)}), \tag{19}$$

for $X \leq x \leq L$. Outside of the specified ranges, the fields are 0. The Hamiltonian for this system is $H_X \otimes H_{L-X}$.

[7]We will take the mirrors to be very massive relative to the total field energy so that their motions induce only higher-order effects in the calculations which follow.

[8]In what follows, we have slightly abused notation by using the same symbol a_n for the annihilation operator acting on $\mathcal{H}_X$ and the annihilation operator acting on $\mathcal{H}_{L-X}$. Similarly with the creation operators. Although symbols should technically be distinguished when acting on different Hilbert spaces, the ambiguity can be entirely resolved by context.

2.1 Vacuum expectation value

The vacuum state of $\mathcal{H}_Z$ is $|0\rangle_Z$ and we write the vacuum state of the tensor product space $\mathcal{H}$ as $|0\rangle_X|0\rangle_{L-X}$. Then the rest-frame vacuum energy density $\mathcal{E}_S(t,x)$ is the vacuum expectation value of the rest-frame Hamiltonian density.

$$\begin{aligned}\mathcal{E}_S(t,x) = \lim_{\epsilon\to 0}[({}_{L-X}\langle 0|)({}_X\langle 0|)\ \frac{1}{2}\{\partial_t\Phi_L^X(t,x+\epsilon)\partial_t\Phi_L^X(t,x)\}_S+ \\ \{\partial_x\Phi_L^X(t,x+\epsilon)\partial_x\Phi_L^X(t,x)\}_S\ \ |0\rangle_X|0\rangle_{L-X}].\end{aligned} \tag{20}$$

It is straightforward to evaluate this expression using $\langle 0|a_n a_n^\dagger|0\rangle = 1$. We get

$$\mathcal{E}_S(t,x) = \lim_{\epsilon\to 0}\begin{cases}\sum_n \frac{\pi n}{2X^2}\cos(\frac{\pi n\epsilon}{X}) & \text{if } 0\le x\le X\\ \sum_n \frac{\pi n}{2(L-X)^2}\cos(\frac{\pi n\epsilon}{(L-X)}) & \text{if } X<x\le L.\end{cases} \tag{21}$$

The total energy is obtained by integrating over the entire region $0\le x\le L$, and taking the $\epsilon\to 0$ limit,[9]

$$\begin{aligned}E_L^X(\epsilon) &= \sum_n \frac{\pi n}{2X}\cos\left(\frac{\pi n\epsilon}{X}\right) + \sum_n \frac{\pi n}{2(L-X)}\cos\left(\frac{\pi n\epsilon}{L-X}\right)\\
&= \frac{d}{d\epsilon}\left\{\frac{1}{2}\sum_n \sin(\pi n\epsilon/X) + \sin(\pi n\epsilon/(L-X))\right\}\\
&= \frac{d}{d\epsilon}\left\{\frac{1}{4i}\sum_n e^{i\pi n\epsilon/X} - e^{-i\pi n\epsilon/X} + e^{i\pi n\epsilon/(L-X)} - e^{-i\pi n\epsilon/(L-X)}\right\}\\
&= \frac{d}{d\epsilon}\left\{\frac{1}{4i}\left[\frac{1}{1-e^{i\pi\epsilon/X}} - \frac{1}{1-e^{-i\pi\epsilon/X}} + \frac{1}{1-e^{i\pi\epsilon/(L-X)}} - \frac{1}{1-e^{-i\pi\epsilon/(L-X)}}\right]\right\}\\
&= \frac{d}{d\epsilon}\left\{\frac{1}{4}[\cot(\pi\epsilon/(2X) + \cot(\pi\epsilon/(2(L-X))]\right\}\\
&= \frac{-\pi}{8}\left[\frac{\csc^2(\pi\epsilon/(2X)}{X} + \frac{\csc^2(\pi\epsilon/(2(L-X)))}{L-X}\right]\\
&= \frac{-L}{2\pi\epsilon^2} - \frac{\pi}{24X} - \frac{\pi}{24(L-X)} + \mathcal{O}(\epsilon).\end{aligned} \tag{22}$$

[9] In the equation which follows, one of the intermediate steps requires the infinite sum of a series like $e^{i\pi n\epsilon/X}$ of terms, each of magnitude 1. Strictly speaking, this series only converges for values of ϵ with a non-zero imaginary part (positive or negative depending on the particular series). However, after taking the sum, the imaginary part can be taken to 0.

This is the familiar 1-D Casimir energy. When L is taken to be very large relative to X, the 3rd term drops out. The first term, which diverges with ϵ, doesn't depend on X and thus doesn't contribute to the force on the inner wall. A more familiar derivation of the Casimir energy can be obtained by observing that the first term above starts with

$$\sum \frac{\pi n}{2X}\cos(\frac{\pi n\epsilon}{X}) = \frac{1}{2}\left[\sum \frac{\pi n}{2X}e^{i\frac{\pi n\epsilon}{X}} + \sum \frac{\pi n}{2X}e^{-i\frac{\pi n\epsilon}{X}}\right]. \tag{23}$$

Each of the sums on the right-hand side are functions of ϵ which can be obtained by analytic continuation[10] to $+i\epsilon$ for the first sum and $-i\epsilon$ for the second sum. In each case, the resulting sum is a heat-kernel regularized sum of mode energies as shown, for example, in Schwartz [5].

Turn now to the $'$-frame, the reference frame moving to the right relative to the rest-frame with speed v. The $'$-frame coordinates x' and t' are related to x and t as in the previous section and the boundaries are at $x' = \gamma(v)(-vt)$, $x' = \gamma(v)(X - vt)$ and $x' = \gamma(v)(L - vt)$. The scalar fields transform as[11]

$$\Phi'_Z(t', x') = \Phi_Z(t, x). \tag{24}$$

So, following eq. (18),

$$\begin{aligned}\partial_{t'}\phi_X(t', x') &= \left(\frac{\partial t}{\partial t'}\partial_t + \frac{\partial x}{\partial t'}\partial_x\right)\phi_X(t, x)\\ &= \sum_n i\sqrt{\frac{n\pi}{X}}\gamma(v)[-i\sin(\frac{n\pi x}{X})(a_n e^{-in\pi t/X} + a_n^\dagger e^{in\pi t/X})\\ &\qquad + v\cos(\frac{n\pi x}{X})(a_n e^{-in\pi t/X} - a_n^\dagger e^{in\pi t/X})],\end{aligned} \tag{25}$$

$$\begin{aligned}\partial_{x'}\phi_X(t', x') &= \left(\frac{\partial t}{\partial x'}\partial_t + \frac{\partial x}{\partial x'}\partial_x\right)\phi_X(t, x)\\ &= \sum_n i\sqrt{\frac{n\pi}{X}}\gamma(v)[-iv\sin(\frac{n\pi x}{X})(a_n e^{-in\pi t/X} + a_n^\dagger e^{in\pi t/X})\\ &\qquad + \cos(\frac{n\pi x}{X})(a_n e^{-in\pi t/X} - a_n^\dagger e^{in\pi t/X})],\end{aligned} \tag{26}$$

and similarly for $\phi_{L-X}(t', x')$.

The vacuum energy density in the $'$-frame is, analogously to eq. (20)

$$\begin{aligned}\mathcal{E}'_S(t', x') = \lim_{\epsilon\to 0}[({}_{L-X}\langle 0|)({}_X\langle 0|)\ \frac{1}{2}(\{(\partial_{t'}\Phi'^X_L(t' - v\gamma(v)\epsilon, x' + \gamma(v)\epsilon)\partial_{t'}\Phi'^X_L(t', x'))\}_S +\\ \{(\partial_{x'}\Phi'^X_L(t' - v\gamma(v)\epsilon, x' + \gamma(v)\epsilon)\partial_{x'}\Phi'^X_L(t', x'))\}_S)\ |0\rangle_X|0\rangle_{L-X}].\end{aligned} \tag{27}$$

[10] Perhaps more rigor would be warranted here to establish analyticity and convergence.

[11] Notice that the subscript Z in this equation is just a label for the fields.

Note that the point-splitting regularization has been obtained by transforming the field arguments from the regularized rest-frame arguments. We will refer to this as "rest-frame regularization". Alternatively, we could have regularized the expression by changing products of the form $\Theta(t', x')\Theta(t', x')$ to the regularized form $\lim_{\epsilon' \to 0} \Theta(t', x' + \epsilon')\Theta(t', x')$, where Θ represents an arbitrary field. We would refer to this as " $'$-frame regularization". Neither procedure is Lorentz covariant and the limits turn out to differ as shall be described shortly. Using the rest-frame regularization as given in eq. (27), integrating the energy density over the contracted regions, and following the same approach as in eqs.(22), we end up with the total energy in the $'$-frame

$$E_L'^X(\epsilon) = \left[\frac{-L}{2\pi\epsilon^2} - \frac{\pi}{24X} + \frac{\pi}{24(L-X)}\right]\gamma(v)(1+v^2) + \mathcal{O}(\epsilon). \tag{28}$$

The expression transforms in just the same (non-vectorial) way that we found in the classical theory. However, had we used $'$-frame regularization, we would have found that the divergent term in the $'$-frame is $-\frac{L}{\gamma(v)2\pi\epsilon'^2}$. The finite part would have been the same as in eq. (28).

2.2 Blackbody radiation

The vacuum expectation value computed in the previous section, corresponds to field theory at 0 temperature. When we examine field theory at finite temperature, we end up computing expectation values of multiparticle states whose single-particle momenta correspond to the modes of the classical theory summarized in the Introduction. It should be mentioned here, that our setup isn't typical of most blackbody analyses. In particular, we have been examining a system which can be regarded as one cavity adjacent to another. The common mirror is reflective on both sides. We arrange that both cavities have the same temperature, and we will analyze the 0-point energy as was done in the previous section. There may be no virtue in analyzing this two-cavity system rather than a single-cavity system, other than for the sake of possible clarity in isolating the 0-point contribution.

Recall that the Hilbert space $\mathcal{H}$ for the theory under consideration, is a tensor product of two Hilbert spaces defined respectively by states localized between the rest positions $0 \le x \le X$, and $X < x \le L$. In each Hilbert space, single-particle states are given by $|n\rangle$ defined in eq. (12). Now generalize the notation to accommodate multi-particle states, defined by

$$\begin{aligned} |m\rangle &\equiv |m_1, m_2, m_3...\rangle \\ &= \frac{a_1^{\dagger m_1}}{\sqrt{m_1!}}\frac{a_2^{\dagger m_2}}{\sqrt{m_2!}}\frac{a_3^{\dagger m_3}}{\sqrt{m_3!}}...|0\rangle, \end{aligned} \tag{29}$$

where the value m_i is the particle occupation number for the i^{th} mode. The full Hilbert space is generated by the tensor products $|m, m'\rangle \equiv |m\rangle_X |m'\rangle_{L-X}$ where the subscripts are used to distinguish the two Hilbert spaces.

The finite-temperature theory can be described by a canonical ensemble, implemented with the canonical density matrix

$$\begin{aligned} \rho_S(T) &= \frac{e^{-\frac{H_S}{k_B T}}}{Z_S} \\ &= \sum_{m,m'} P(m, m', T)|m, m'\rangle\langle m, m'|, \end{aligned} \tag{30}$$

where the partition function is $Z_S = \mathrm{Tr}\left(e^{-\frac{H_S}{k_B T}}\right)$ and the probability P is derived as follows. From eq. (14) which specifies the Hamiltonian density, and from eqs. (18) and (19) for the fields, we obtain the rest Hamiltonian density

$$\tilde{H}_S(t,x) = \lim_{\epsilon\to 0} \begin{cases} \sum_{n,m} \frac{\pi\sqrt{nm}}{2X^2}\{[\sin(\frac{\pi n(x+\epsilon)}{X})\sin(\frac{\pi m x}{X}) + \cos(\frac{\pi n(x+\epsilon)}{X})\cos(\frac{\pi m x}{X})] \\ (a_n^\dagger a_m e^{\frac{i(n-m)\pi t}{X}} + a_n a_m^\dagger e^{-\frac{i(n-m)\pi t}{X}}) + \\ [\sin(\frac{\pi n(x+\epsilon)}{X})\sin(\frac{\pi m x}{X}) - \cos(\frac{\pi n(x+\epsilon)}{X})\cos(\frac{\pi m x}{X})](a_n^\dagger a_m^\dagger e^{\frac{i(n+m)\pi t}{X}} + a_n a_m e^{-\frac{i(n+m)\pi t}{X}})\} \\ \text{if } 0 \le x \le X \\ \sum_{n,m} \frac{\pi\sqrt{nm}}{2(L-X)^2}\{[\sin(\frac{\pi n(x+\epsilon)}{L-X})\sin(\frac{\pi m x}{L-X}) + \cos(\frac{\pi n(x+\epsilon)}{L-X})\cos(\frac{\pi m x}{L-X})] \\ (a_n^\dagger a_m e^{\frac{i(n-m)\pi t}{L-X}} + a_n a_m^\dagger e^{-\frac{i(n-m)\pi t}{L-X}}) + \\ [\sin(\frac{\pi n(x+\epsilon)}{L-X})\sin(\frac{\pi m x}{L-X}) - \cos(\frac{\pi n(x+\epsilon)}{L-X})\cos(\frac{\pi m x}{L-X})](a_n^\dagger a_m^\dagger e^{\frac{i(n+m)\pi t}{L-X}} + a_n a_m e^{-\frac{i(n+m)\pi t}{L-X}})\} \\ \text{if } 0 \le x \le (L-X). \end{cases} \tag{31}$$

When this expression is integrated to obtain the rest-frame Hamiltonian, the terms with $a_n^\dagger a_m^\dagger$ and $a_n a_m$ for all m and n, as well as terms $a_n^\dagger a_m$ and $a_n a_m^\dagger$ for $n \neq m$, all integrate to 0. The remaining term simplifies by invoking a simple trigonometric identity and rewriting $a_N a_N^\dagger$ as $a_N^\dagger a_N + 1$. The result is

$$H_S = \lim_{\epsilon\to 0} \sum_n \left\{ \frac{\pi n}{2X}\cos(\frac{\pi n \epsilon}{X})(2a_N^\dagger a_N + 1) + \frac{\pi n}{2(L-X)}\cos(\frac{\pi n\epsilon}{L-X})(2a_N^\dagger a_N + 1) \right\}. \tag{32}$$

Recalling that $a_n^\dagger a_n |m\rangle_X = m_n |m\rangle_X$ (and similarly for $|m\rangle_{L-X}$) we see that the eigenvectors and eigenvalues of H_S are given by

$$H_S|m, m'\rangle = E_{m,m'}|m, m'\rangle, \tag{33}$$

where

$$
\begin{aligned}
E_{m,m'} = [\lim_{\epsilon\to 0}\sum_n \frac{\pi n m_n}{X}\cos(\frac{\pi n\epsilon}{X}) + \frac{\pi n m'_n}{L-X}\cos(\frac{\pi n\epsilon}{L-X})+ \\
\sum_n \frac{\pi n}{2X}\cos(\frac{\pi n\epsilon}{X}) + \sum_n \frac{\pi n}{2(L-X)}\cos(\frac{\pi n\epsilon}{(L-X)})].
\end{aligned}
\tag{34}
$$

The second sum is familiar from the first line of eq. (22) as the Casimir contribution. The term $P(m,m',T)$ which appears in eq. (30) is seen to be

$$
P(m,m',T) = \frac{e^{-\beta E_{m,m'}}}{Z_S}, \tag{35}
$$

where $\beta = \frac{1}{k_B T}$, k_B is Boltzmann's constant and the partition function is

$$
\begin{aligned}
Z_S &= \sum_{m,m'} e^{-\beta E_{m,m'}} \\
&= \sum_{m,m'}\prod_n e^{-\beta\cos(\frac{\pi n\epsilon}{X})\frac{\pi n(m_n+\frac{1}{2})}{X}} \prod_{n'} e^{-\beta\cos(\frac{\pi n'\epsilon}{L-X})\frac{\pi n'(m_{n'}+\frac{1}{2})}{X}} \\
&= \prod_n\sum_m e^{-\beta\cos(\frac{\pi n\epsilon}{X})\frac{\pi n(m_n+\frac{1}{2})}{X}} \prod_{n'}\sum_{m'} e^{-\beta\cos(\frac{\pi n'\epsilon}{L-X})\frac{\pi n'(m_{n'}+\frac{1}{2})}{L-X}} \\
&= \prod_{j=1,j'=1}^{\infty,\infty} \frac{e^{-\beta\frac{j\pi\cos(\frac{j\pi\epsilon}{X})}{2X}}}{(1-e^{-\beta\frac{j\pi}{X}\cos(\frac{j\pi\epsilon}{X})})} \frac{e^{-\beta\frac{j'\pi\cos(\frac{j'\pi\epsilon}{L-X})}{2(L-X)}}}{(1-e^{-\beta\frac{j'\pi}{L-X}\cos(\frac{j'\pi\epsilon}{L-X})})}.
\end{aligned}
\tag{36}
$$

We can now compute the ensemble average of the energy. This is given by $\mathrm{Tr}(\rho(T)H_S)$ and follows the same approach as the derivation of Z_S.

$$
\begin{aligned}
\mathrm{Tr}[\rho(T)H_S] &= \frac{1}{Z_S}\sum_{m,m'} E_{m,m'}e^{-\beta E^{m,m'}} \\
&= -\frac{\partial\log(Z_S)}{\partial\beta} \\
&= \lim_{\epsilon\to 0} E_{L,T}(\epsilon),
\end{aligned}
\tag{37}
$$

where

$$E_{L,T}(\epsilon) = \sum_j \left[\frac{j\pi \cos(\frac{j\pi\epsilon}{X})}{2X} \left(1 + \frac{2e^{-\beta\frac{j\pi}{X}cos(\frac{j\pi\epsilon}{X})}}{1-e^{-\beta\frac{j\pi}{X}cos(\frac{j\pi\epsilon}{X})}} \right) + \frac{j\pi \cos(\frac{j\pi\epsilon}{L-X})}{2(L-X)} \left(1 + \frac{2e^{-\beta\frac{j\pi}{L-X}cos(\frac{j\pi\epsilon}{L-X})}}{1-e^{-\beta\frac{j\pi}{L-X}cos(\frac{j\pi\epsilon}{L-X})}} \right) \right]$$
$$= \left[\sum_j \left(\frac{j\pi}{X}\right) \frac{e^{-\beta\frac{j\pi}{X}}}{1-e^{-\beta\frac{j\pi}{X}}} + \left(\frac{j\pi}{L-X}\right) \frac{e^{-\beta\frac{j\pi}{L-X}}}{1-e^{-\beta\frac{j\pi}{L-X}}} \right]$$
$$+ \frac{-L}{2\pi\epsilon^2} - \frac{\pi}{24X} - \frac{\pi}{24(L-X)} + \mathcal{O}(\epsilon), \tag{38}$$

where the last line of this equation was previously derived in eq. (22). The first line and part of the second can be expressed in terms of the Dedekind eta function $\eta(\tau)$ so altogether

$$E_{L,T}(\epsilon) = \frac{d}{d\beta} \ln \left[\eta(\frac{i\beta}{2X})\eta(\frac{i\beta}{2(L-X)}) \right] - \frac{L}{2\pi\epsilon^2} + \mathcal{O}(\epsilon). \tag{39}$$

Now let us derive the energy in the $'$-frame. The density-matrix basis will be taken to be the same in the $'$-frame as in the rest frame. One important difference in the $'$-frame, is that the Hamiltonian contains non-zero terms of the form $a_n^\dagger a_m$ where $n \neq m$ as well as non-zero terms of the form $a_n a_m$ and $a_n^\dagger a_m^\dagger$ for all n, m. In the rest frame, those terms are absent because in the scalar field operator, such terms are multiplied by trigonometric products of the form $\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})$. Since $m \neq n$, those trigonometric products integrate to 0. However, in the $'$-frame, the change of coordinates results in an integral which looks like, for example,

$$F(n,m) =$$
$$\int_k^{\frac{L}{\gamma(v)}+k} dx \sin(\frac{n\pi\gamma(v)(x+vt)}{L}) \sin(\frac{m\pi\gamma(v)(x+vt)}{L}) cos\frac{(n-m)\pi\gamma(v)(t+vx)}{L}. \tag{40}$$

This integral is generally neither 0 nor time-independent. A similar phenomenon is derived in Celmaster [2] for the classical electromagnetic field where the $'$-frame Hamiltonian oscillates in time[12].

In order to proceed with the computation of the $'$-frame finite temperature energy, we note that the quantity of interest to us will be

$$\mathrm{Tr}[H'_S\rho(T)] = \mathrm{Tr}[H'_S e^{-\beta H_S}]/Z_S$$
$$= \sum_{m,m'} \langle m, m'|H'_S e^{-\beta H_S}|m, m'\rangle/Z_S. \tag{41}$$

[12]This fact already illustrates that the energy and momentum don't transform as a Lorentz 4-vector

The reason that we employ the same density matrix in the $'$-frame as in the rest frame, is that the ensemble probabilities of basis states are assumed to be frame-independent[13]. Consider those terms in the Hamiltonian of the type above with $a_n^\dagger a_m$ $(n \neq m)$. We see that for those terms and products of those terms with those of H_S, we get $\langle m, m'|a_n^\dagger a_m ...|m, m'\rangle = 0$. Similarly with all terms of the form $a_n a_m$ and $a_n^\dagger a_m^\dagger$.

If we define $\hat{H}'_S$ as H'_S minus all terms of the form $a_n^\dagger a_m$ $(n \neq m)$ and all terms of the form $a_n a_m$ and $a_n^\dagger a_m^\dagger$, then $|m, m'\rangle$ are eigenvalues of $\hat{H}'_S$ and by above, we see that

$$\mathrm{Tr}[H'_S \rho(T)] = \mathrm{Tr}[\hat{H}'_S \rho(T)]. \tag{42}$$

We obtain $\hat{H}'_S$ by first Lorentz-transforming, as shown in eqs. (25) and (26), $\phi_X(t, x)$ and $\phi_{L-X}(t, x)$. Following the same approach used to obtain equation (28) for the $'$-frame vacuum energy we find that that the $'$-frame $\hat{H}_S$ eigenvalues corresponding to the rest-frame values given in equation (34) are

$$E'_{m,m'} = \gamma(v)(1 + v^2) E_{m,m'}. \tag{43}$$

Then the $'$-frame ensemble average of the energy is

$$\begin{aligned} \mathrm{Tr}(\rho(T) H'_S) &= \frac{1}{Z_S} \sum_{m,m'} E'_{m,m'} e^{-\beta E_{m,m'}} \\ &= \gamma(v)(1 + v^2) \frac{1}{Z_S} \sum_{m,m'} E_{m,m'} e^{-\beta E_{m,m'}} \\ &= \lim_{\epsilon \to 0} \gamma(v)(1 + v^2) E_{L,T}(\epsilon), \end{aligned} \tag{44}$$

where the rest-frame average energy $E_{L,T}(\epsilon)$ is given by equation (39). Once again, the energy transforms non-vectorially just as in the classical theory.

3 Quantum Electrodynamics

The 3-D treatment of radiation in a cavity, closely follows what was done in the previous section for a 1-D scalar field. The enclosure to be considered is a perfectly reflective right parallelepiped cavity with coordinates $(0 < x_1 < L_1, 0 < x_2 < L_2, 0 < x_3 < X)$ contiguous to a perfectly reflective right

[13] It would be difficult to imagine a notion of probability which is frame-dependent. Nevertheless some frame must be chosen in which we comfortably can invoke the principle of equal a priori probabilities. The rest frame seems the natural choice hence the one we use here for the density matrix.

parallelepiped cavity with coordinates ($0 < x_1 < L_1, 0 < x_2 < L_2, X < x_3 < L$). (Alternatively, think of this as a single cavity whose third dimension is of length L but which has a partition located at $x_3 = X$ and which is reflective on both of its sides.)

The electric and magnetic field operators within each cavity are

$$E^j_{L_3}(t,\mathbf{x}) = \sum''_{\mathbf{n}\equiv(n_1,n_2,n_3),\alpha} 2\sqrt{\frac{\omega(\mathbf{n},L_3)}{L_1L_2L_3}}\frac{1}{\tan(\frac{\pi n_j x_j}{L_j})}\prod_{k=1}^{3}\sin(\frac{\pi n_k x_k}{L_k}) \\ [ia_{\mathbf{n},\alpha}e^j_\alpha(\mathbf{n},L_3)e^{-i\omega(\mathbf{n},L_3)t} - ia^\dagger_{\mathbf{n},\alpha}e^j_\alpha(\mathbf{n},L_3)e^{i\omega(\mathbf{n},L_3)t}],$$

$$B^j_{L_3}(t,\mathbf{x}) = \sum''_{\mathbf{n}\equiv(n_1,n_2,n_3),\alpha} 2\sqrt{\frac{1}{\omega(\mathbf{n},L_3)L_1L_2L_3}}\tan(\frac{\pi n_j x_j}{L_j})(\prod_{k=1}^{3}\cos(\frac{\pi n_k x_k}{L_k})) \\ \varepsilon^{jkl}\frac{\pi n_k}{L_k}e_{l,\alpha}(\mathbf{n},L_3)[a_{\mathbf{n},\alpha}e^{-i\omega(\mathbf{n},L_3)t} + a^\dagger_{\mathbf{n},\alpha}e^{i\omega(\mathbf{n},L_3)t}]. \tag{45}$$

The double prime on the summation symbol denotes that in the sum, $n_i \geq 0$, at most one of the n_i equals 0 and in that case, there is only one polarization instead of 2.[14] In these expressions, the parameter L_3 denotes the x_3 dimension of the cavity, so has the value X or $L - X$. Also,

$$\omega(\mathbf{n},L_3) \equiv \sqrt{\frac{\pi^2 n_1^2}{L_1^2} + \frac{\pi^2 n_2^2}{L_2^2} + \frac{\pi^2 n_3^2}{L_3^2}}. \tag{46}$$

As usual, $\hbar$ is set to 1, the $a_{\mathbf{n},\alpha}$ and $a^\dagger_{\mathbf{n},\alpha}$ are lowering and raising operators on the Hilbert Space $\mathcal{H}_C$ with commutation relations $[a_{\mathbf{n},\alpha}, a^\dagger_{\mathbf{n}',\alpha'}] = \delta_{\mathbf{n},\mathbf{n}'}\delta_{\alpha,\alpha'}$, and the $\mathbf{e}_\alpha(\mathbf{n},L_3)$ are Coulomb gauge polarization vectors with

$$\mathbf{e}_\alpha(\mathbf{n},L_3)\cdot\mathbf{n} = 0, \\ \sum_{\alpha=1}^{2} e_{i,\alpha}(\mathbf{n},L_3)e_{j,\alpha}(\mathbf{n},L_3) = \delta_{ij} - \frac{\pi^2 n_i n_j}{L_iL_j\omega^2(\mathbf{n},L_3)}. \tag{47}$$

The Hilbert Space $\mathcal{H}_C$ denotes the Fock space generated by one-particle states of the form

$$|\mathbf{n},\alpha\rangle = a^\dagger_{\mathbf{n},\alpha}|0\rangle. \tag{48}$$

In a way similar to what was done with the 1-D scalar field, we can write

$$\mathbf{E}^X_L(t,\mathbf{x}) = \mathbf{E}_X(t,\mathbf{x})\otimes\mathbf{E}_{L-X}(t,\mathbf{x}), \\ \mathbf{B}^X_L(t,\mathbf{x}) = \mathbf{B}_X(t,\mathbf{x})\otimes\mathbf{B}_{L-X}(t,\mathbf{x}). \tag{49}$$

[14] The single polarization also leads to the correctly normalized commutation relations between the vector potential and the electric field.

where $\mathbf{E}_L^X(t,\mathbf{x})$ and $\mathbf{B}_L^X(t,\mathbf{x})$ operate on the Hilbert space

$$\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_{L-X}. \tag{50}$$

It would seem natural, in analogy to the 1-D theory, to write a regulated version of the rest-frame Hamiltonian density as

$$\bar{H}_{EM}^{\epsilon}(t,\mathbf{x}) = \lim_{\epsilon\to 0}\frac{1}{2}[\{\mathbf{E}_L^X(t,\mathbf{x}+\boldsymbol{\epsilon})\cdot\mathbf{E}_L^X(t,\mathbf{x})\}_S + \{\mathbf{B}_L^X(t,\mathbf{x}+\boldsymbol{\epsilon})\cdot\mathbf{B}_L^X(t,\mathbf{x})\}_S], \tag{51}$$

where as before, the notation $\{AB\}_S$ denotes the symmetrized product $\frac{1}{2}(AB+BA)$. Unfortunately, the vector $\boldsymbol{\epsilon}$ breaks rotational invariance and the $\boldsymbol{\epsilon}\to 0$ limit could lead to a wrong Casimir force. Further discussion about frame-dependent regularization can be found, for example, in DeWitt [6] and Hagen [7]. In lieu of equation (51) we will therefore write

$$\tilde{H}_{EM}(t,\mathbf{x}) = \lim_{\Lambda\to\infty}\frac{1}{2}\mathcal{R}_\Lambda[\mathbf{E}_L^X(t,\mathbf{x})\cdot\mathbf{E}_L^X(t,\mathbf{x}) + \mathbf{B}_L^X(t,\mathbf{x})\cdot\mathbf{B}_L^X(t,\mathbf{x})]. \tag{52}$$

where $\mathcal{R}_\Lambda$ denotes a Gaussian regulator such as that described in Schwartz [5]. The total rest-frame Hamiltonian is

$$H_{EM} = \int_0^{L_1} dx_1 \int_0^{L_2} dx_2 \int_0^{L} dx_3 \tilde{H}_{EM}(t,\mathbf{x}). \tag{53}$$

Expanding the integrand, inserting the regulator and then integrating, we obtain

$$H_{EM} = \lim_{\Lambda\to\infty} H_{EM}^{\Lambda}, \tag{54}$$

where

$$\begin{aligned} H_{EM}^{\Lambda} &= \frac{1}{2}\sum_{\mathbf{n},\alpha}{}' \omega(\mathbf{n},X)e^{-(\frac{\omega(\mathbf{n},X)}{\pi\Lambda})^2}[a_{\mathbf{n},\alpha}a_{\mathbf{n},\alpha}^\dagger + a_{\mathbf{n},\alpha}^\dagger a_{\mathbf{n},\alpha}]_X \\ &+\frac{1}{2}\sum_{\mathbf{n},\alpha}{}' \omega(\mathbf{n},L-X)e^{-(\frac{\omega(\mathbf{n},L-X)}{\pi\Lambda})^2}[a_{\mathbf{n},\alpha}a_{\mathbf{n},\alpha}^\dagger + a_{\mathbf{n},\alpha}^\dagger a_{\mathbf{n},\alpha}]_{L-X}. \end{aligned} \tag{55}$$

The prime on the summation symbol denotes that in the sum, $n_i \geq 0$, at most one of the n_i equals 0 and in that case, the term should be multiplied by $\frac{1}{2}$.[15] The notation $[...]_C$ surrounding annihilation and creation operators, indicates that those operators are acting on the Hilbert space $\mathcal{H}_C$. There are

[15] The prime is used in treatments such as Milloni [8] and Casimir [9]. Those authors observe that when one of the n_i is 0, photons have only one, rather than two polarizations, thus an energy contribution half as large as for other modes. This was accounted for in eq.

no terms proportional to $a_{\mathbf{n},\alpha}a_{\mathbf{n},\alpha}$ or $a^\dagger_{\mathbf{n},\alpha}a^\dagger_{\mathbf{n},\alpha}$ because their coefficients (when adding the electric and magnetic contributions) are each zero. H^Λ_{EM} can be simplified somewhat by invoking the commutation relations. This leads to an expression somewhat similar to H in eq. (32).

$$\begin{aligned} H^\Lambda_{EM} &= \sum_{\mathbf{n},\alpha}{}' \omega(\mathbf{n},X) e^{-(\frac{\omega(\mathbf{n},X)}{\pi\Lambda})^2} \left([a^\dagger_{\mathbf{n},\alpha}a_{\mathbf{n},\alpha}]_X + \frac{1}{2} \right) \\ &+ \sum_{\mathbf{n},\alpha}{}' \omega(\mathbf{n},L-X) e^{-(\frac{\omega(\mathbf{n},L-X)}{\pi\Lambda})^2} \left([a^\dagger_{\mathbf{n},\alpha}a_{\mathbf{n},\alpha}]_{L-X} + \frac{1}{2} \right). \end{aligned} \tag{56}$$

From this, we can compute the energy vacuum expectation value as

$$\begin{aligned} E_{0,EM} &= ({}_{L-X}\langle 0|) ({}_X\langle 0|) \quad H_{EM} \quad |0\rangle_X |0\rangle_{L-X} \\ &= \lim_{\Lambda\to\infty} E^\Lambda_{0,EM}, \end{aligned} \tag{57}$$

where

$$E^\Lambda_{0,EM} = \left\{ \sum_{\mathbf{n}}{}' \left[\omega(\mathbf{n},X) e^{-(\frac{\omega(\mathbf{n},X)}{\pi\Lambda})^2} \right] + \sum_{\mathbf{n}}{}' \left[\omega(\mathbf{n},L-X) e^{-(\frac{\omega(\mathbf{n},L-X)}{\pi\Lambda})^2} \right] \right\}. \tag{58}$$

For simplicity, take $L_1 = L_2 = L$. Employing a Euler-MacLaurin formula (see, for example, Abramowitz and Stegun [[10]]) and following the derivation of Casimir [9] (and also derived later in this paper) we obtain

$$E^\Lambda_{0,EM} = \frac{\pi^2\sqrt{\pi}}{8} L^3\Lambda^4 - \frac{L^2\pi^2}{720X^3} \left[1 + \mathcal{O}(\frac{X}{L}, \frac{1}{X\Lambda}) \right]. \tag{59}$$

Only the X-dependent term is of physical significance.

---

(45) via the double-prime notation. This, however, would appear to be cancelled by the energy contribution required for the integrals in 3-space, as shown in eq. (53). Specifically, for modes where all $n_i \neq 0$, the integrands are of the form $tr_1^2(\frac{n_1\pi x_1}{L_1})tr_2^2(\frac{n_2\pi x_2}{L_2})tr_3^2(\frac{n_3\pi x_3}{L_3})$ where the $tr_i()$ are either cos() or sin() and each term with $tr_i^2()$ integrates to a value of $\frac{L_i}{2}$. The 3-D integral is therefore equal $\frac{L_1L_2L_3}{8}$. However, when $n_i = 0$ for some value of i, the integral is 0 unless $tr_i()$ is cos() in which case that factor is 1, and the integral over that dimension is L_i. The 3-D integral is then $\frac{L_1L_2L_3}{4}$ which is twice the value for modes where all $n_i \neq 0$. In particular, for modes where one of the $n_i = 0$, that factor of two exactly cancels the factor of two coming from the fact that those modes have only a single polarization. Nevertheless, there is a remaining factor of 2 which arises from the fact that when one of the $n_i = 0$, then only one component of the electric mode and two components of the magnetic mode are non-zero. When performing the appropriate sums of products of polarization vectors, it turns out that these add up to a half of what they would if all electromagnetic mode components were non-zero. That factor of $\frac{1}{2}$ then leads to our definition of the primed sum.

We turn now to blackbody radiation for electromagnetism. This analysis closely follows what was done in the previous section, for the 1-D scalar theory. Begin by defining multi-particle states analogously to the 1-D definition of equation (29)

$$\begin{aligned}|\{m,\alpha\}> &\equiv |m_{(1,1)}, m_{(1,2)}, m_{(2,1)}, m_{(2,2)}... > \\ &= \frac{(\hat{a}^{\dagger}_{\mathbf{k}^{\mathbf{n}_1},1})^{m_{1,1}}}{\sqrt{m_{1,1}!}} \frac{(\hat{a}^{\dagger}_{\mathbf{k}^{\mathbf{n}_1},2})^{m_{1,2}}}{\sqrt{m_{1,2}!}} \frac{(\hat{a}^{\dagger}_{\mathbf{k}^{\mathbf{n}_2},1})^{m_{2,1}}}{\sqrt{m_{2,1}!}} \frac{(\hat{a}^{\dagger}_{\mathbf{k}^{\mathbf{n}_2},2})^{m_{2,2}}}{\sqrt{m_{2,2}!}} ...|0>,\end{aligned} \tag{60}$$

where $m_{i,j}$ is the occupancy number for mode i with polarization j. The full Hilbert space is generated by the tensor products $|\{m,\alpha\},\{m',\alpha'\}\rangle \equiv |\{m,\alpha\}\rangle_X |\{m',\alpha'\}\rangle_{L-X}$ where the subscripts are used to distinguish the two Hilbert spaces.

The canonical density matrix, similar to that in equation (30) is

$$\begin{aligned}\rho_{EM}(T) &= \frac{e^{-\frac{H_{EM}}{k_B T}}}{Z_{EM}} \\ &= \sum_{\{m,\alpha\},\{m',\alpha'\}} P_{EM}(\{m,\alpha\},\{m',\alpha'\},T)|\{m,\alpha\},\{m,\alpha'\}\rangle\langle\{m,\alpha\},\{m',\alpha'\}|,\end{aligned} \tag{61}$$

with the partition function $Z_{EM} = \mathrm{Tr}(e^{-\frac{H_{EM}}{k_B T}})$ and $P_{EM}(\{m,\alpha\},\{m',\alpha'\},T)$ to be derived similarly to the derivation of $P(m,m',T)$ in equation (35)). We begin the derivation of P_{EM} by noting from equation (56), that the eigenvalues of H_{EM} are

$$\begin{aligned}E_{\{m,\alpha\},\{m',\alpha'\}} = \lim_{\Lambda\to\infty}[&{\sum_{\mathbf{n},\alpha}}' m_{\mathbf{n},\alpha}\omega(\mathbf{n},X)e^{-(\frac{\omega(\mathbf{n},X)}{\pi\Lambda})^2} + {\sum_{\mathbf{n},\alpha}}' m'_{\mathbf{n},\alpha}\omega(\mathbf{n},L-X)e^{-(\frac{\omega(\mathbf{n},L-X)}{\pi\Lambda})^2} \\ &+ {\sum_{\mathbf{n},\alpha}}' \frac{1}{2}\omega(\mathbf{n},X)e^{-(\frac{\omega(\mathbf{n},X)}{\pi\Lambda})^2} + {\sum_{\mathbf{n},\alpha}}' \frac{1}{2}\omega(\mathbf{n},L-X)e^{-(\frac{\omega(\mathbf{n},L-X)}{\pi\Lambda})^2}].\end{aligned} \tag{62}$$

Therefore

$$P_{EM}(\{m,\alpha\},\{m',\alpha'\},T) = \frac{e^{-\beta E_{\{m,\alpha\},\{m',\alpha'\}}}}{Z_{EM}}, \tag{63}$$

with

$$Z_{EM} = \lim_{\Lambda\to\infty} Z^{\Lambda}_{EM}, \tag{64}$$

where

$$
\begin{aligned}
Z_{EM}^{\Lambda} &= \sum_{\{m,\alpha\},\{m',\alpha'\}} [\prod_{\mathbf{n},\alpha}{}' e^{-\beta m_{\mathbf{n},\alpha}\omega(\mathbf{n},X)} \prod_{\mathbf{n},\alpha}{}' e^{-\beta m'_{\mathbf{n},\alpha}\omega(\mathbf{n},L-X)} \\
&\quad \prod_{\mathbf{n},\alpha}{}' e^{\frac{-\beta}{2}\omega(\mathbf{n},X)e^{-(\frac{\omega(\mathbf{n},X)}{\pi\Lambda})^2}} \prod_{\mathbf{n},\alpha}{}' e^{-\frac{\beta}{2}\omega(\mathbf{n},L-X)e^{-(\frac{\omega(\mathbf{n},L-X)}{\pi\Lambda})^2}}] + \mathcal{O}(\frac{1}{\Lambda}) \\
&= \prod_{\mathbf{n}}{}'(1-e^{-\beta\omega(\mathbf{n},X)})^{-2} \prod_{\mathbf{n}}{}'(1-e^{-\beta\omega(\mathbf{n},L-X)})^{-2} \\
&\quad \prod_{\mathbf{n}}{}' e^{-\beta\omega(\mathbf{n},X)e^{-(\frac{\omega(\mathbf{n},X)}{\pi\Lambda})^2}} \prod_{\mathbf{n}}{}' e^{-\beta\omega(\mathbf{n},L-X)e^{-(\frac{\omega(\mathbf{n},L-X)}{\pi\Lambda})^2}}] + \mathcal{O}(\frac{1}{\Lambda}).
\end{aligned}
\tag{65}
$$

The prime on the product symbol denotes that in the product, $n_i \geq 0$, at most one of the n_i equals 0 and in that case, the square-root should be taken for that term. Notice that the first two products don't involve the Gaussian regularization term. This is because its effect on those terms (which turn out to converge for finite Λ) is $\mathcal{O}(\frac{1}{\Lambda})$.

The thermal average energy is

$$
\begin{aligned}
\langle H_{EM}\rangle &= -\frac{\partial \log(Z_{EM})}{\partial\beta} \\
&= \sum_{\mathbf{n}}{}'(\frac{2\omega(\mathbf{n},X)e^{-\beta\omega(\mathbf{n},X)}}{1-e^{-\beta\omega(\mathbf{n},X)}} + \frac{2\omega(\mathbf{n},L-X)e^{-\beta\omega(\mathbf{n},L-X)}}{1-e^{-\beta\omega(\mathbf{n},L-X)}})+ \\
&\quad \lim_{\Lambda\to\infty}[\sum_{\mathbf{n}}{}'\omega(\mathbf{n},X)e^{-(\frac{\omega(\mathbf{n},X)}{\pi\Lambda})^2} + \sum_{\mathbf{n}}{}'\omega(\mathbf{n},L-X)e^{-(\frac{\omega(\mathbf{n},L-X)}{\pi\Lambda})^2}].
\end{aligned}
\tag{66}
$$

We recognize the last line as the large-Λ limit of eq. (58). Taking $L_1 = L_2 = L$ and the large L and large Λ limits, we obtain (some details of this calculation are provided later, when we examine the $'$-frame) the usual Stefan-Boltzmann law plus Casimir contribution.

$$
\langle H_{EM}\rangle = \lim_{\Lambda\to\infty}\left\langle H_{EM}^{\Lambda}\right\rangle, \tag{67}
$$

where

$$
\left\langle H_{EM}^{\Lambda}\right\rangle = \frac{L^3\pi^2}{15\beta^4}(1+\mathcal{O}(\frac{1}{L}, e^{-\beta/X})) + \frac{\pi^2\sqrt{\pi}}{8}L^3\Lambda^4 - \frac{L^2\pi^2}{720X^3}[1+\mathcal{O}(\frac{X}{L},\frac{1}{X\Lambda})]. \tag{68}
$$

The limits considered in this expression are $L \to \infty$ and $\beta \to \infty$. The asymptotic behavior $\mathcal{O}(e^{-\beta/X})$ denotes that the term is exponentially small as $\beta/X \to \infty$. For comparison, see Schwinger *et al.*[11].

The ′-frame field energy can also be computed and the steps are similar to those described for the 1-D scalar field theory. The electric and magnetic field components of eq. (45) transform in the ′-frame according to eqs. (4) and (5). Contributions to the thermal average of the ′-frame Hamiltonian, come from terms proportional to either the identity or the number operators $a^\dagger_{\mathbf{n}} a_{\mathbf{n}}$. Terms proportional to v integrate to zero. The remaining parts of the Hamiltonian can then be assembled either by direct calculations with the polarization vectors, or with identities of the kind employed for the analogous classical calculation in McDonald [1] and Celmaster [2]. The result is that each mode is multiplied by $\gamma(v)\left(1+\frac{n_3^2 v^2\pi^2}{\omega^2(\mathbf{n},L_3)L_3^2}\right)$ where L_3 is either X or $L-X$ depending on context. The thermal average of the ′-frame Hamiltonian becomes[16]

$$
\begin{aligned}
\langle H'_{EM}\rangle_T = & \sum_{\mathbf{n}}{}' \gamma(v)\left(1+\frac{n_3^2v^2\pi^2}{\omega^2(\mathbf{n},X)X^2}\right)\frac{2\omega(\mathbf{n},X)e^{-\beta\omega(\mathbf{n},X)}}{1-e^{-\beta\omega(\mathbf{n},X)}}+ \\
& \sum_{\mathbf{n}}{}' \gamma(v)\left(1+\frac{n_3^2v^2\pi^2}{\omega^2(\mathbf{n},L-X)(L-X)^2}\right)\frac{2\omega(\mathbf{n},L-X)e^{-\beta\omega(\mathbf{n},L-X)}}{1-e^{-\beta\omega(\mathbf{n},L-X)}}+ \\
& \lim_{\Lambda\to\infty}[\sum_{\mathbf{n}}{}' \gamma(v)\left(1+\frac{n_3^2v^2\pi^2}{\omega^2(\mathbf{n},X)X^2}\right)\omega(\mathbf{n},X)e^{-(\frac{\omega(\mathbf{n},X)}{\pi\Lambda})^2}+ \\
& \sum_{\mathbf{n}}{}' \gamma(v)\left(1+\frac{n_3^2v^2\pi^2}{\omega^2(\mathbf{n},L-X)(L-X)^2}\right)\omega(\mathbf{n},L-X)e^{-(\frac{\omega(\mathbf{n},L-X)}{\pi\Lambda})^2}].
\end{aligned} \tag{69}
$$

The summation of terms appears to have somewhat different consequences for the ′-frame Stefan-Boltzmann energy E'_{SB}, and the ′-frame Casimir energy E'_C. Both summations will be done with a Euler-MacLaurin (E-M) expansion [10]. First we consider the Stefan-Boltzmann contribution. The n_1 and n_2 summations are done first. Then the second line of eq. (69) is, for $L_1 = L_2 = L$, and to leading order in L,

$$
E_{SB}^{\prime(L-X)} = \frac{L^2}{\pi^2}\sum_{n_3}{}'\int dxdy\gamma(v)\sqrt{x^2+y^2+h^2n_3^2}(1+\frac{v^2h^2n_3^2}{x^2+y^2+h^2n_3^2})\frac{e^{-\beta\sqrt{x^2+y^2+h^2n_3^2}}}{1-e^{-\beta\sqrt{x^2+y^2+h^2n_3^2}}}, \tag{70}
$$

where $h=\frac{\pi}{L-X}$ and the superscript $L-X$ refers to the second line of equation (69). Now invoking the E-M expansion for the n_3 sum, this becomes to

[16] Note that we don't choose to change the regularization despite the fact that coordinates have been Lorentz-transformed.

leading order in h,

$$E_{SB}^{'(L-X)} \approx \left(\frac{L(L-X)}{\pi^3}\right) \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} d\theta \sin(\theta) \int_0^{\infty} drr^2 \left[r(1+v^2\cos(\theta)) \frac{e^{-r}}{1-e^{-r}} \right]$$
$$= \gamma(v) \left(1+\frac{v^2}{3}\right) E_{SB}^{L-X}. \tag{71}$$

As usual, unprimed quantities refer to the rest frame. A similar result holds for the first line of eq. (69) , and this is therefore suppressed by $\mathcal{O}(\frac{1}{L})$ so that altogether

$$E'_{SB} = \gamma(v) \left(1+\frac{v^2}{3}\right) E_{SB} \left[1+\mathcal{O}(\frac{1}{L})\right]. \tag{72}$$

For the $'$-frame Casimir energy, start with the third line of eq. (69). Define $\tilde{h} = \frac{1}{X\Lambda}$. As above, the sums over n_1 and n_2 become integrals over x_1 and x_2, and to leading order in L we obtain,

$$E_C^{'X} = \pi L^2 \Lambda^3 \sum_{n_3}{}' \int dxdy\gamma(v) \sqrt{x^2+y^2+\tilde{h}^2 n_3^2} \left(1+\frac{v^2\tilde{h}^2 n_3^2}{x^2+y^2+\tilde{h}^2 n_3^2}\right) e^{-(x^2+y^2+\tilde{h}^2 n_3^2)}. \tag{73}$$

Following Casimir [9], change the x, y integral to an integral over polar coordinates r and ϕ, then define

$$F(z) = \pi L^2 \Lambda^3 \gamma(v) \int_0^{\frac{\pi}{2}} d\phi \int_0^{\infty} drr\sqrt{r^2+z^2}(1+\frac{v^2 z^2}{r^2+z^2})e^{-(r^2+z^2)}, \tag{74}$$

and change coordinates so that $u = r^2 + z^2$. Then,

$$F(z) = \frac{\pi^2 L^2 \Lambda^3}{4} \gamma(v) \int_{z^2}^{\infty} du\sqrt{u}(1+\frac{v^2 z^2}{u})e^{-u}. \tag{75}$$

The E-M expansion requires the following derivatives of F, evaluated at $z = 0$ and $z = \infty$. Set $\tilde{K} = \frac{\pi^2 L^2 \Lambda^3 \gamma(v)}{4}$.

$$F'(z) = \tilde{K} \left\{ v^2\sqrt{\pi} \left[-2z\mathrm{erf}(z) - \frac{2z^2 e^{-z^2}}{\sqrt{\pi}} + 2z \right] - 2z^2 e^{-z^2} \right\}.$$
$$F''(z) = \tilde{K} \left\{ v^2\sqrt{\pi} \left[-2\mathrm{erf}(z) + \frac{4z^3 e^{-z^2}}{\sqrt{\pi}} - \frac{8z e^{-z^2}}{\sqrt{\pi}} + 2 \right] + e^{-z^2} \left[4z^3 - 4z \right] \right\}.$$
$$F'''(z) = \tilde{K} e^{-z^2} \left\{ v^2 \left[-8z^4 + 28z^2 - 12 \right] - 8z^4 + 20z^2 - 4 \right\}. \tag{76}$$

The E-M expansion [10] for $E_C^{'X}$ is (noting that because of the exponential factor, F and all of the derivatives are 0 when evaluated at $z = \infty$)

$$E_C^{'X} = \frac{1}{\tilde{h}} \int_0^\infty dz F(z) - \frac{1}{2} F(0) + \frac{1}{2} F(0) - \tilde{h} \frac{B_2}{2} F^{'}(0) - \tilde{h}^3 \frac{B_4}{24} F^{'''}(0) + \mathcal{O}(\tilde{h}^5), \tag{77}$$

where the Bernoulli numbers have the values $B_2 = \frac{1}{6}$ and $B_4 = -\frac{1}{30}$. The first term is most easily evaluated starting with the expression for F given in eq. (74) and converting the r, z integral into polar coordinates.

$$\begin{aligned} \frac{1}{\tilde{h}} \int_0^\infty dz F(z) &= \pi L^2 X \Lambda^4 \gamma(v) \int_0^{\frac{\pi}{2}} d\phi \int_0^\infty dz \int_0^\infty dr r \sqrt{r^2 + z^2} (1 + \frac{v^2 z^2}{r^2 + z^2}) e^{-(r^2+z^2)} \\ &= \frac{\pi^2}{2} L^2 X \Lambda^4 \gamma(v) \int_0^{\frac{\pi}{2}} d\theta \int_0^\infty ds s^2 \cos\theta (1 + v^2 \sin^2\theta) e^{-s^2} \\ &= \frac{\pi^2 \sqrt{\pi}}{8} L^2 X \Lambda^4 \gamma(v) (1 + \frac{v^2}{3}). \end{aligned} \tag{78}$$

The second term of Eq. (77) arises because of the effect of the prime superscript in the summation sign of eq. (73). The remaining terms add up to $-(\frac{1}{X\Lambda})^3 \frac{1}{720} (\tilde{K} \frac{\pi}{4} \gamma(v) [4(1 + v^2) + 4v^2 \ + 4v^2] + \mathcal{O}(\tilde{h}^5)$. A similar result holds for the last line of equation (69) but substituting $L - X$ for X.

We now have all of the components of $\langle H'_{EM} \rangle_T$.

$$\langle H'_{EM} \rangle_T = \lim_{\Lambda \to \infty} \left\langle H_{EM}^{'\Lambda} \right\rangle_T, \tag{79}$$

where

$$\left\langle H_{EM}^{'\Lambda} \right\rangle_T = \gamma(v) \left\{ \frac{L^3 \pi^2}{15 \beta^4} \left(1 + \frac{v^2}{3} \right) + \frac{\pi^2 \sqrt{\pi}}{8} L^3 \Lambda^4 \left(1 + \frac{v^2}{3} \right) - \frac{L^2 \pi^2}{720 X^3} (1 + 3v^2) + ... \right\}. \tag{80}$$

The ellipsis denotes terms that are subleading, relative to the others, in $\frac{1}{L}$ and e^{-L}. Once again, we see that the energy of confined radiation does not transform as a component of a Lorentz 4-vector. The nature of the transformation differs from term to term. The Stefan-Boltzmann energy is multiplied by $(1 + \frac{v^2}{3})$ which is an average of the mode factor obtained classically in eq. (7). The divergent term (proportional to Λ^4) transforms the same way, but as observed in the discussion following eq. (28), this may be an artifact of how we transform the regularization when going from the rest-frame to the $'$-frame. By contrast to the Stefan-Boltzmann term, the leading order (finite part of) the Casimir energy is multiplied by $(1 + 3v^2)$.[17]

[17] In the discussion following eq. (28), it was also observed that for the two regulariza-

Summary

It has been shown previously in [1], [2] that in a rectangular box with perfectly reflecting walls, the electromagnetic energy does not transform as a component of a 4-vector[18]. In this paper, the analysis was extended to quantum fields. We first looked at a one-dimensional scalar field theory, and found that the average thermal energy of the confined field, transforms identically to what we found in the classical theory of confined particles, $E' = \gamma(v)(1+v^2)E$ where E' is the energy in a frame moving at velocity v relative to the rest frame. This result holds for individual modes, and also for both ordinary blackbody radiation and for the Casimir effect. Things are somewhat different for Quantum Electrodynamics. There, we see that the non-vectorial factor is different for blackbody (Stefan-Bolzmann) radiation, which has a correction factor of $(1 + \frac{v^2}{3})$ than for the (finite) Casimir energy which has a correction factor of $(1+3v^2)$. The observation of the blackbody non-vectorial transformation is not new, although it has tended to be buried in literature concerning anisotropy of the cosmic microwave background (CMB). The initial papers ([12], [13], [14], [15]) pertaining to motion within a black body, were primarily focused on the directional radiation which had been detected in CMB experiments. Results have tended to be expressed in terms of effective temperatures rather than energies, although the work by Ford and O'Connell [16] has helped clarify the relationship between derivations. All of these authors implicitly or explicitly recognize that the directional asymmetry is a consequence of the fact that the CMB dictates a preferred frame of reference. However, the papers cited do not illustrate the precise local nature of the Lorentz transformations as expressed in the previous sections.

Acknowledgments

This work began with a query, posed by Dale Landis and Paul Jameson, on how classical electromagnetic energy in a cavity, transforms in a moving reference frame. I had expected to demonstrate the usual Lorentz transformation of an energy-momentum vector and found instead that there is an

tions considered, both led to the same finite term. However, that is not a general proof that sensible Lorentz non-covariant regularizations should always lead to the same finite Casimir terms. It is generally held that this should be the case, but an analysis done by Hagen [7] demonstrates that the issue is, at best, subtle.

[18] To be precise, we should say "The EM-field energy and momentum do not transform as a 4-vector". However, in our analyses we always start with the rest frame (where the 3-momentum is 0) and therefore it suffices to examine only the transformed energy to see that the Lorentz transformation is not vectorial.

interplay with the cavity walls, which effectively modifies the Lorentz transformation as discussed above. I then turned to the quantum theory and discovered that what should have been an entirely straightforward exercise in Quantum Electrodynamics, ended up exposing a number of assumptions and details that I had been sweeping under the proverbial rug. I'm grateful to Matt Schwartz for allowing me to attend his excellent class on Quantum Field Theory, and whose text [5] has been a constant companion. I'd also like to thank Nick Agia, who helped me find a normalization error that was driving me crazy and in the process showed me some cool identities.

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