Alternate statement of LSZ theorem

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The purpose of this exercise is to demonstrate two equivalent formulations of the LSZ theorem. We'll do this by taking an example of the LSZ theorem (the same one given in my notes on scattering) and working with it.

The statement in my scattering notes was:

$$\begin{aligned}
&\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = \\
&\quad [i\int d^4x_1 e^{-ip_1^{in}\cdot x_1}(\Box_{x_1}+m^2)][i\int d^4x_2 e^{-ip_2^{in}\cdot x_2}(\Box_{x_2}+m^2)] \\
&\quad [i\int d^4x_3 e^{+ip_1^{out}\cdot x_3}(\Box_{x_3}+m^2)][i\int d^4x_4 e^{+ip_2^{out}\cdot x_4}(\Box_{x_4}+m^2)] \\
&\quad \mathcal{G}(x_1,x_2,x_3,x_4)
\end{aligned} \tag{1}$$

where the D'Alembertian $\Box_{\mathbf{x}}$ is defined by $\Box_x \equiv \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$. This LSZ example is easily generalized to more ingoing and outgoing particles with multiple masses and associated with other quantum fields.

The alternative formulation, which is to proven below, is

$$\begin{aligned}
&\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = \\
&\quad (-(p_1^{out})^2+m^2)(-(p_2^{out})^2+m^2)(-(p_1^{in})^2+m^2)(-(p_2^{in})^2+m^2) \\
&\quad [i\int d^4x_1 e^{-ip_1^{in}\cdot x_1}][i\int d^4x_2 e^{-ip_2^{in}\cdot x_2}][i\int d^4x_3 e^{+ip_1^{out}\cdot x_3}][i\int d^4x_4 e^{+ip_2^{out}\cdot x_4}] \\
&\quad \mathcal{G}(x_1,x_2,x_3,x_4)
\end{aligned} \tag{2}$$

- In the first part of this exercise we'll look at a much-simplified version of the RHS of equation (1). Define $f(p)$ by

$$f(p) = \int_{-\infty}^{+\infty} dx e^{-ipx}\partial_x g(x) \tag{3}$$

where for now, x and p are 1-dimensional. Recall that ∂_x is defined to be the derivative with respect to x (in other words, $\frac{d}{dx}$). Now assume that the function $g(x)$ drops rapidly to 0 when $x \to \pm\infty$.

- Prove that

$$f(p) = ip \int_{-\infty}^{+\infty} dx e^{-ipx} g(x) \tag{4}$$

Hint: Use 'integration by parts', and if you don't remember how to do that, check wikipedia for a quick review.

- Now prove that

$$\int_{-\infty}^{+\infty} dx e^{-ipx} \partial_x^2 g(x) = -p^2 \int_{-\infty}^{+\infty} dx e^{-ipx} g(x) \tag{5}$$

Hint: In equation (3), substitute $g(x)$ with $\partial_x g(x)$, then apply equation (4) first to $\partial_x g$ and then again to g.

- Finally, prove that

$$\int_{-\infty}^{+\infty} dx e^{-ipx} (\partial_x^2 + m^2) g(x) = (-p^2 + m^2) \int_{-\infty}^{+\infty} dx e^{-ipx} g(x) \tag{6}$$

• We're almost ready to tackle the full RHS equation (1). Let's do this in two pieces.

  - First, extend equation (6) to 4 dimensions. We simply let x and p now be 4-vectors, and in the exponent, we'll have $-ip \cdot x$ where $p \cdot x = x_0 p_0 - x_1 p_1 - x_2 p_2 - x_3 p_3$. We'll also replace ∂_x^2 by ∂^2 which is defined by $\partial_{x_0}^2 - \partial_{x_1}^2 - \partial_{x_2}^2 - \partial_{x_3}^2$.

    Prove that

$$\int_{-\infty}^{+\infty} d^4x e^{-ip\cdot x} (\partial^2 + m^2) g(x) = (-p^2 + m^2) \int_{-\infty}^{+\infty} d^4x e^{-ip\cdot x} g(x) \tag{7}$$

    where as usual, p^2 means $p \cdot p$.

    Hint: Simply write out the integral as $\int dx_0 \int dx_1 \int dx_2 \int dx_3$ and apply equation (6) to each integral.

– The RHS of equation (1) is

$$[i\int d^4x_1 e^{-ip_1^{in}\cdot x_1}(\Box_{x_1}+m^2)][i\int d^4x_2 e^{-ip_2^{in}\cdot x_2}(\Box_{x_2}+m^2)]$$
$$[i\int d^4x_3 e^{+ip_1^{out}\cdot x_3}(\Box_{x_3}+m^2)][i\int d^4x_4 e^{+ip_2^{out}\cdot x_4}(\Box_{x_4}+m^2)]$$
$$\mathcal{G}(x_1,x_2,x_3,x_4)$$

where the D'Alembertian $\Box_{\mathbf{x}}$ is defined by $\Box_x \equiv \partial_t^2-\partial_x^2-\partial_y^2-\partial_z^2$. Notice that the notation may be a bit confusing relative to the notation we've used in previous equations. Now I have the variables x_1, etc. **each** referring to a 4-vector, and for each of those 4-vectors, I have denoted the components as (t,x,y,z) instead of the components used in previous equations – namely (x_0,x_1,x_2,x_3). If nothing else, the value of this exercise is for you to sort all that out! Once you've done this, prove that the above expression equals

$$(-(p_1^{out})^2+m^2)(-(p_2^{out})^2+m^2)(-(p_1^{in})^2+m^2)(-(p_2^{in})^2+m^2)$$
$$[i\int d^4x_1 e^{-ip_1^{in}\cdot x_1}][i\int d^4x_2 e^{-ip_2^{in}\cdot x_2}][i\int d^4x_3 e^{+ip_1^{out}\cdot x_3}][i\int d^4x_4 e^{+ip_2^{out}\cdot x_4}]$$
$$\mathcal{G}(x_1,x_2,x_3,x_4) \tag{8}$$

Hint: Apply equation (7) to each of the 4 arguments.

- All of this might seem like a lot of warm-up. However, it's all just been a pretty straightforward application of Fourier transform technology. What we've done is to rewrite the LSZ theorem equation (2)