Very simple proof of the Central Limit Theorem

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This doesn't attempt to show limits or convergence or any of the other things required for rigour. However, the method appears reasonable and since the answer is correct, it's probably straightforward to make the argument rigorous.

Let $P(x)$ be a probability distribution with mean 0 (for simplicity) and standard deviation σ. Define

$$\mathcal{P}_{lim}(x) = \lim_{N\to\infty} \int dx_1 ... dx_N P(x_1)...P(x_N) \delta(x - \frac{(x_1 + ... + x_N)}{N}) \tag{1}$$

We want to show that $\mathcal{P}_{lim}(x)$ is a Gaussian probability distribution with mean 0 and standard deviation that goes as $\frac{\sigma}{\sqrt{N}}$.[1]

Write the δ function in equation (1) as

$$\delta(x - \frac{(x_1 + ... + x_N)}{N}) = \frac{1}{2\pi} \int dt e^{it(x - \frac{(x_1+...+x_N)}{N})} \tag{2}$$

and then equation (1) becomes

$$\begin{aligned} \mathcal{P}_{lim}(x) &= \frac{1}{2\pi} \lim_{N\to\infty} \int dt e^{itx} dx_1 ... dx_N P(x_1) e^{\frac{itx_1}{N}} ... P(x_N) e^{\frac{itx_1}{N}} \\ &= \frac{1}{2\pi} \lim_{N\to\infty} \int dt e^{itx} [\mathcal{K}(t, N)]^N \end{aligned} \tag{3}$$

[1] This is a good example of muddy notation. Rather than define $\mathcal{P}_{lim}(x)$ as a limit, it would be better to define it as some function of N with a leading term in N and corrections that vanish relative to the leading term, as $N \to \infty$.

where

$$\begin{aligned}\mathcal{K}(t,N) &= \int dx P(x) e^{\frac{itx}{N}} \\ &= \int dx P(x)(1 + \frac{itx}{N} - \frac{1}{2}(\frac{tx}{N})^2 + \mathcal{O}(\frac{1}{N^3})) \\ &= (1 + \frac{1}{N}\mathcal{Z}(t,N))\end{aligned} \tag{4}$$

with

$$\begin{aligned}\mathcal{Z}(t,N) &= \frac{-t^2}{2N}\int dx x^2 P(x) \\ &= \frac{-t^2}{2N}\sigma^2\end{aligned} \tag{5}$$

In the last line of (4), we have used the fact that $\int dx P(x) = 1$. In (5), we've used the facts that the term linear in x is proportional to the mean, and that the mean has a value of 0 by assumption.

Finally, returning to (3),(4), and (5) and using the continuous compounding formula $\lim_{N\to\infty}(1+\frac{\alpha}{N})^N = e^{\alpha}$, we get[1]

$$\begin{aligned}\mathcal{P}_{lim}(x) &= \frac{1}{2\pi}\int dt e^{itx} \lim_{N\to\infty}[1 + \frac{\mathcal{Z}(t,N)}{N}]^N \\ &= \frac{1}{2\pi}\int dt e^{itx} e^{\frac{-t^2}{2N}\sigma^2} \\ &= \frac{1}{\sqrt{2\pi}\frac{\sigma}{\sqrt{N}}} e^{\frac{-x^2}{\sigma/\sqrt{N}}}\end{aligned} \tag{6}$$

This is what was to be proved.