Scattering and cross-sections

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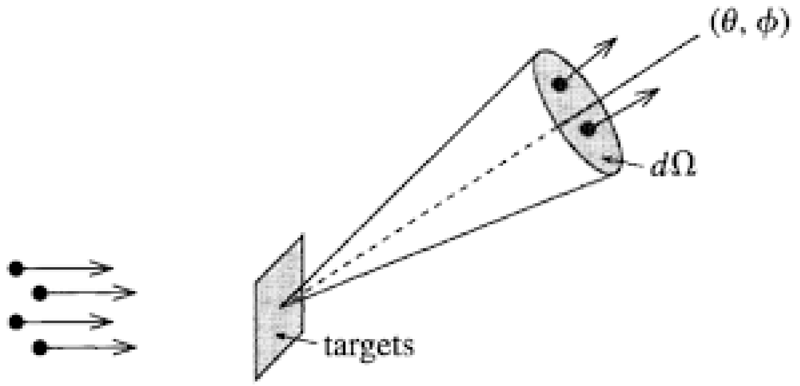
1 Cross-sections

- The chief tools for testing theories of elementary particles are *scattering experiments*.
- A beam of particles is aimed at either a stationary target or at a beam coming head-on.
- After collision, particles *scatter* in all directions with all energies (conserving total energy and momentum).

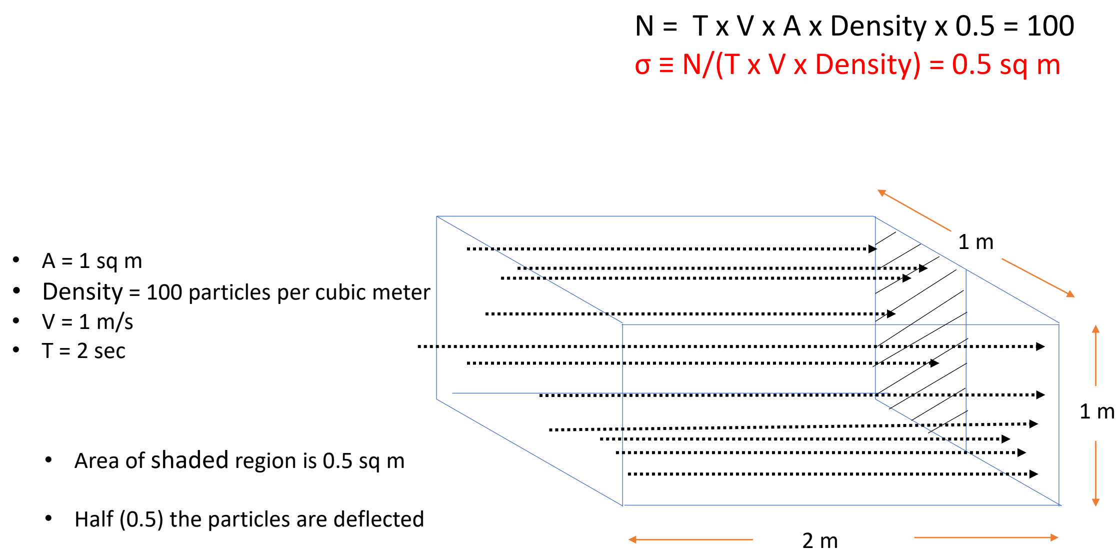


- Many particles go straight through. The number of deflected particles is proportional to the *cross section*.
  - This is denoted as σ.
  - Or we can count the number that go through the solid angle $d\Omega$. This is proportional to the *differential cross section* and is denoted as $\frac{d\sigma}{d\Omega}$.

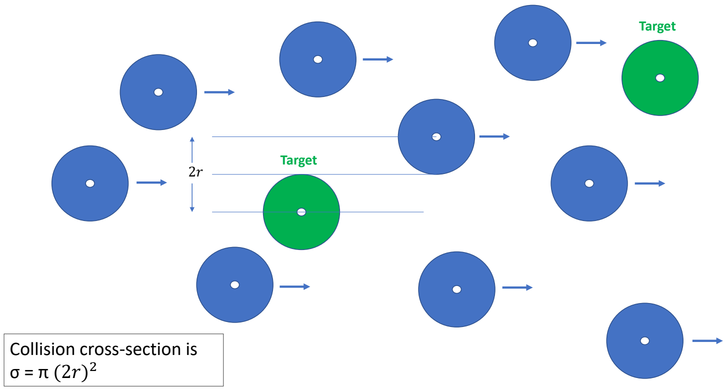
- Definition of cross-section (area)

$$\sigma = \frac{\text{number of particles scattered}}{\text{time x number density in beam x velocity of beam}}$$

The figure below serves as a sanity check. The shaded region on the RH wall deflects particles and the unshaded region lets them through. So only 0.5 m^2 of surface deflects particles. The above equation correctly gives the cross-section as 0.5 m^2.

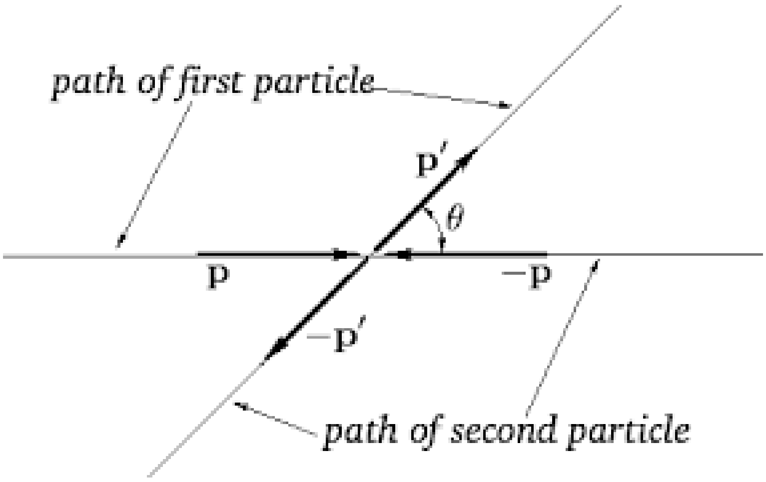


- The cross section measures the effective strength of interaction and can be computed from first principles.



2 Scattering

- The simplest scattering experiment involves two colliding identical particles.



Ingoing particles

$$p_1^{in} = (E, \vec{p})$$
$$p_2^{in} = (E, -\vec{p})$$

Outgoing particles

$$p_1^{out} = (E', \vec{p}')$$
$$p_2^{out} = (E', -\vec{p}')$$

- Question: What is the probability amplitude that if you start with the above ingoing particles as shown, then you'll end up with the above outgoing particles?

Quantum mechanics answer is

$$\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle$$

where S is the operator propagating *in* to *out* states. S is called the **S matrix**.

- **Our goal in quantum field theory is to compute S!** We can do this using the path integral. Remember, the cross-section σ is proportional to the total number of deflected particles, so it must be related to S which gives probabilities that particles are deflected a certain amount.

  In general,

  $$\sigma = \int d^4p_1^{out} d^4p_2^{out} f(p_1^{out}, p_2^{out}) |\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle|^2$$

  where the function f is an appropriate weighting of the final states, and σ is a function of the initial states.

- Example: Electron colliding with positron and ending up with muon and anti-muon.

  If you compute a low-energy approximation to the S-matrix from the path integral, you get

  $$|\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle|^2 \propto 1 + \cos^2\theta$$

  provided $\theta \neq 0$. Then, for non-zero θ

  $$\frac{d\sigma}{d\Omega} \propto \frac{(1 + \cos^2\theta)}{E^2}$$

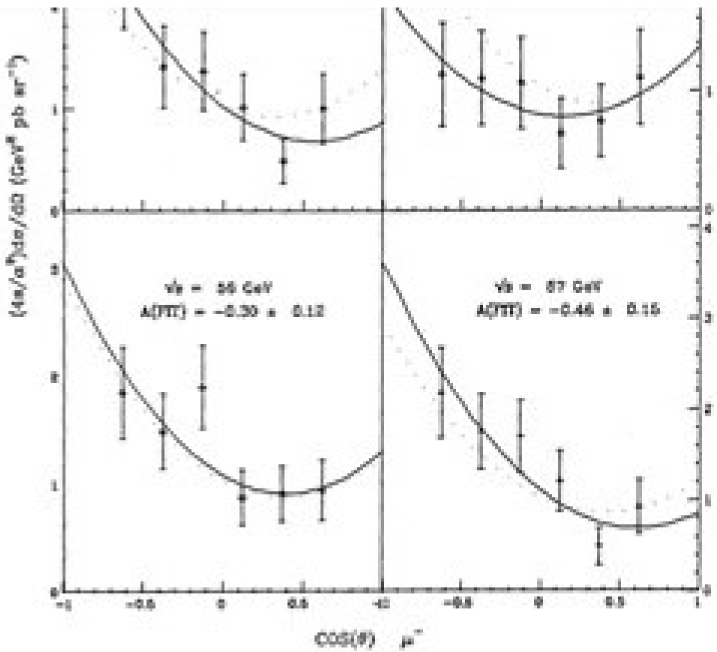


Figure 7.1: The angular distribution for $e^+e^- \to \mu^+\mu^-$ at various c.m. energies.

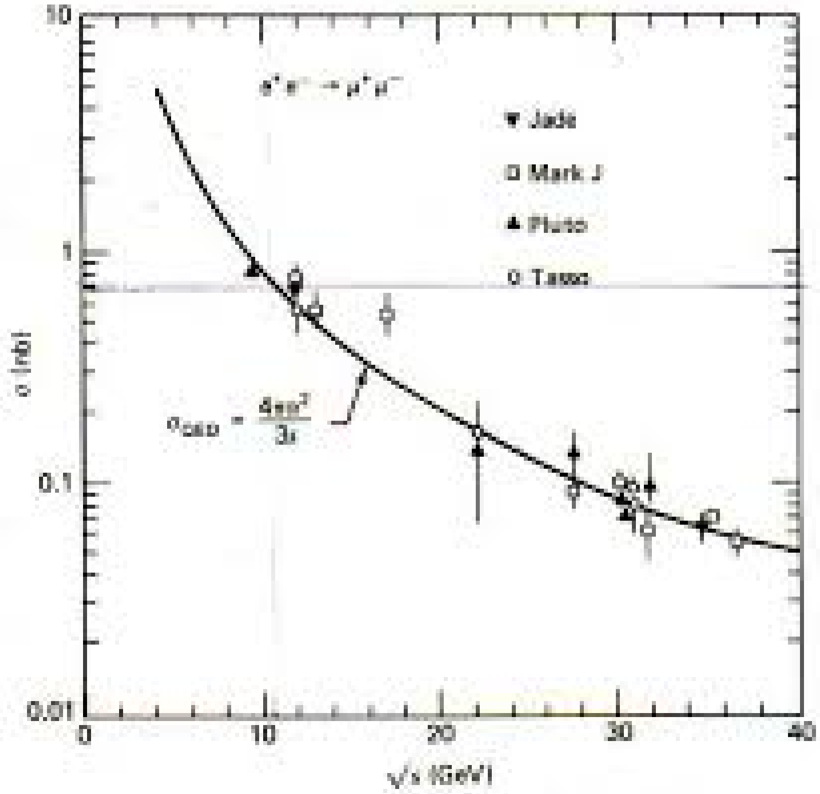


Fig. 6.6 The total cross section for $e^- e^+ \to \mu^- \mu^+$ measured at PETRA versus the center-of-mass energy.

3 Scattering Matrix and Green Functions

- **LSZ Theorem example.**

  Assume that the initial and final particles have mass m and are obtained from the quantum field $\phi(x)$. Then (non-trivial proof) !!!

$$\begin{aligned}
&\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = \\
&\quad (-(p_1^{out})^2 + m^2)(-(p_2^{out})^2 + m^2)(-(p_1^{in})^2 + m^2)(-(p_2^{in})^2 + m^2) \\
&\quad [i \int d^4x_1 e^{-ip_1^{in}\cdot x_1}][i \int d^4x_2 e^{-ip_2^{in}\cdot x_2}][i \int d^4x_3 e^{+ip_1^{out}\cdot x_3}][i \int d^4x_4 e^{+ip_2^{out}\cdot x_4}] \\
&\quad \mathcal{G}(x_1, x_2, x_3, x_4)
\end{aligned} \tag{1}$$

- This LSZ example is easily generalized to more ingoing and outgoing particles with multiple masses and associated with other quantum fields.

- **FROM THIS, THE S-MATRIX CAN BE EASILY COMPUTED FROM THE GREEN FUNCTIONS.**

4 Philosophizing about Scattering

- We've shown that cross-sections and the S-matrix can be computed from Green functions.

  Earlier, we showed that **Green functions can all be computed from the path integral**.

  The path integral is convenient because you can temporarily forget about operators.

- Question: Can we infer all fundamental physics from the S-matrix and therefore from the path integral?

  Answer: Maybe. If so, then we should be able to infer the Lagrangian from a full knowledge of all scattering experiments.

- This is a very popular point of view in Quantum Field Theory. BUT ...

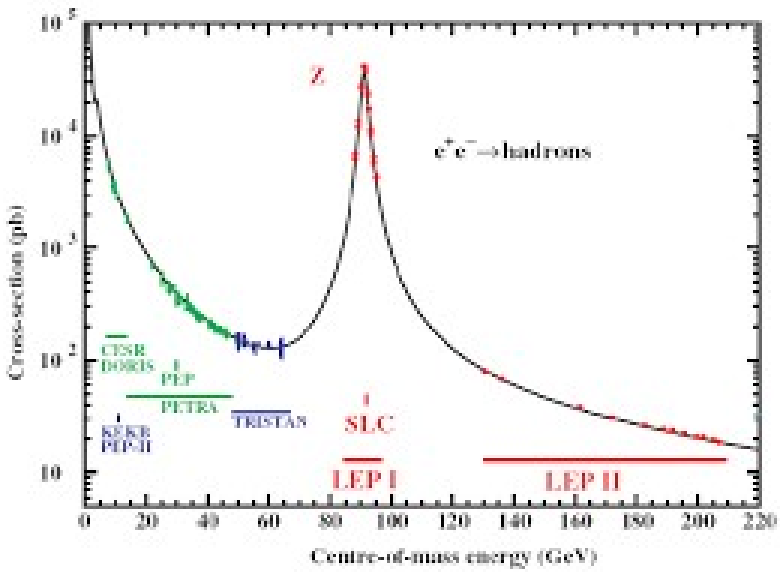
  - Some things may be easier to figure out some other way. For example, the weight of a piece of gold can be measured on a scale, and can be predicted using basic chemistry.
  - Some scattering experiments are impractical or too hard. For example, we can't do experiments at the energies available during the first microseconds of the universe, but we can infer physics from non-scattering calculations in cosmology.

- Practically speaking, there are two complementary approaches to quantum field theory.

  - Path integrals and Green functions which make direct contact to the S-matrix.
  - Hilbert Space, Lagrangians of operator-valued fields, and the canonical commutation relations. These make direct contact to the energy eigenstates a.k.a. the particle spectrum.

- The two approaches are intertwined. In particular, the connection of Green functions to the S-matrix, can only be proven (the LSZ theorem) by using the Hilbert-space/canonical-commutation approach.

- But you can also learn about the particle spectrum from Green functions and scattering experiments. For example, suppose we collide an electron and a positron. If the energy of collision is larger than about 91 GeV, then this would be enough energy to create a Z particle. This particle is likely to be unstable and then could decay, for example, into other particles. What happens in our electron-positron scattering experiment?



The S-matrix at these higher energies becomes

$$|\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle|^2 \propto \frac{E^2}{[(4E^2 - M_Z^2)]^2} g(\theta, ...)$$

which blows up when the centre-of-mass energy $2E$ is equal to $M_Z =$ 91 GeV.[1]

[1] $g(\theta)$ can be computed but isn't pertinent for understanding the peak in the total cross-section. Also, you'll see that the graph describes an experiment where the end particles aren't muons. However, it turns out the peak-feature of the cross-section doesn't depend on the end states.