What is the Dirac Equation?

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Outline

- Lorentz covariance of the wave equation
- Lorentz non-covariance of the Schrodinger equation
- Finding a Lorentz covariant first order differential equation (Dirac equation)
- Things to worry about
  - Consistency of transformations
  - Adding a mass term
- Dirac notation

Wave equation: One space dimension

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = 0 \qquad \text{(in units where } c = 1\text{)}$$

Consider the transformation (Lorentz transformation)

$$x'(x,t) = \gamma(v)(x + vt) \ \ldots \text{ where } \gamma(v) = \frac{1}{\sqrt{1-v^2}}$$

$$t'(x,t) = \gamma(v)(t + vx)$$

$$\varphi'(x',t') = \varphi(x,t)$$

← Observer moving at speed v

Objective: Demonstrate that $\frac{\partial^2 \varphi'}{\partial t'^2} - \frac{\partial^2 \varphi'}{\partial x'^2} = 0.$

When the equation looks the same after transformation, we will say the equation is Lorentz covariant.

Apply the Chain Rule

$$\frac{\partial\varphi(x,t)}{\partial t} = \frac{\partial\varphi'\big(x'(x,t),t'(x,t)\big)}{\partial t}$$

$$= \frac{\partial\varphi\prime}{\partial x\prime}\frac{\partial x\prime}{\partial t} + \frac{\partial\varphi\prime}{\partial t\prime}\frac{\partial t\prime}{\partial t}$$

$$= \frac{\partial\varphi\prime}{\partial x\prime}\gamma(v)v + \frac{\partial\varphi\prime}{\partial t\prime}\gamma(v)$$

$$\varphi'(x',t') = \varphi(x,t)$$

$$x'(x,t) = \gamma(v)(x+vt)$$
$$t'(x,t) = \gamma(v)(t+vx)$$

Do it again

$$\frac{\partial^2\varphi(x,t)}{\partial t^2} = 2\frac{\partial^2\varphi'(x,t)}{\partial t'\partial x'}(\gamma^2(v)v) + \frac{\partial^2\varphi'(x,t)}{\partial x'^2}(\gamma^2(v)v^2) + \frac{\partial^2\varphi'(x,t)}{\partial t'^2}(\gamma^2(v))$$

Similarly

$$-\frac{\partial^2\varphi(x,t)}{\partial x^2} = -2\frac{\partial^2\varphi'(x,t)}{\partial t'\partial x'}(\gamma^2(v)v) - \frac{\partial^2\varphi'^{(x,t)}}{\partial x'^2}(\gamma^2(v)) - \frac{\partial^2\varphi'(x,t)}{\partial t'^2}(\gamma^2(v)v^2)$$

Add

$$\frac{\partial^2\varphi(x,t)}{\partial t^2} - \frac{\partial^2\varphi(x,t)}{\partial x^2} = \gamma^2(v)(1-v^2)\left(\frac{\partial^2\varphi'^{(x,t)}}{\partial t'^2} - \frac{\partial^2\varphi'(x,t)}{\partial x'^2}\right)$$

$$= \frac{\partial^2\varphi'^{(x,t)}}{\partial t'^2} - \frac{\partial^2\varphi\prime(x,t)}{\partial x'^2} \quad \textbf{QED}$$

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}}$$

Schrodinger Equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2}$$

Lorentz transformation as usual, with $\psi'(x', t') = \psi(x, t)$

$$i\frac{\partial\psi'}{\partial t'}\gamma(v) + i\frac{\partial\psi'}{\partial x'}v\gamma(v) = -\frac{1}{2m}[\frac{\partial^2\psi'}{\partial t'\partial x'}\gamma^2(v)v + \frac{\partial^2\psi'}{\partial x'^2}\gamma^2(v) + \frac{\partial^2\psi\prime}{\partial t'^2}\gamma^2(v)v^2]$$

The new equation does **NOT** look the same. **Not covariant**

Find a covariant equation linear in time

Can't do it unless there are at least two components $\psi^0(x,t),\ \psi^1(x,t)$

Use vector notation $\psi(x,t) \equiv \begin{pmatrix} \psi^0(x,t) \\ \psi^1(x,t) \end{pmatrix}$ and $\frac{\partial \psi}{\partial t} = \begin{pmatrix} \frac{\partial \psi^0}{\partial t} \\ \frac{\partial \psi^1}{\partial t} \end{pmatrix}$ etc.

Now GUESS!

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial \psi}{\partial t} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial \psi}{\partial x} = 0$$

This is an example of the Dirac equation.

Is it covariant?

How does ψ transform?

Recall $x'(x,t) = \gamma(v)(x+vt)$ … where $\gamma(v) = \frac{1}{\sqrt{1-v^2}}$

$$t'(x,t) = \gamma(v)(t+vx)$$

We also had to say how the field (or wavefunction) transforms when we change coordinates. For a single component it was easy. For multiple components it's trickier because implicitly we need to be imagining some way to measure each component. For example, if you hold up a yardstick, but rotate the coordinate system, you'll get different x, y and z components describing the yardstick.

GUESS

$$\psi'^0(x',t') = \frac{\sqrt{1-v}}{\sqrt{1+v}}\psi^0(x,t)$$

$$\psi'^1(x',t') = \frac{\sqrt{1+v}}{\sqrt{1-v}}\psi^1(x,t)$$

Success

- Start with the Dirac equation and write it out in components

$$\frac{\partial \psi^1}{\partial t} - \frac{\partial \psi^1}{\partial x} = 0$$

$$\frac{\partial \psi^0}{\partial t} + \frac{\partial \psi^0}{\partial x} = 0$$

- Use the chain rule and the transformation law for ψ.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial \psi\prime}{\partial t\prime} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial \psi\prime}{\partial x\prime} = 0$$

This has the same form as the original Dirac equation. So it is covariant.

Things to worry about #1: transformations?

The ψ transformation law seems arbitrary. Is it?

Observer 2 moving at speed v' from viewpoint of observer 1

Observer 1 moving at speed v from viewpoint of observer 0 at rest

Observer 2 moves at speed v'' relative to observer 0. **Relativity says $v'' = \frac{v'+v}{1+v'v}$**

Remember $\psi'^0(x',t') = \frac{\sqrt{1-v}}{\sqrt{1+v}}\psi^0(x,t)$. So $\psi''^0(x'',t'') = \frac{\sqrt{1-v'}}{\sqrt{1+v'}}\psi'^0(x',t')$.

But if transformation law is consistent, then also $\psi''^0(x'',t'') = \frac{\sqrt{1-v''}}{\sqrt{1+v''}}\psi^0(x,t)$. Is it?

Easy to verify that $\frac{\sqrt{1-v''}}{\sqrt{1+v''}} = \frac{\sqrt{1-v'}}{\sqrt{1+v'}}\frac{\sqrt{1-v}}{\sqrt{1+v}}$ so the transformation law is a consistent "Lorentz group representation"

Things to worry about #2: Mass term

Our Dirac equation did not have a mass term. A natural guess for an equation with mass term is (recall that ψ is a vector)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial\psi}{\partial t}+\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial\psi}{\partial x}+im\psi \;=0$$

Rewrite this as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial\psi}{\partial t}+\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial\psi}{\partial x}+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}im\psi \;=0$$

Now perform a Lorentz transformation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial\psi\prime}{\partial t\prime}+\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial\psi\prime}{\partial x\prime}+\begin{pmatrix} 1-v & 0 \\ 0 & 1+v \end{pmatrix}\frac{im}{\sqrt{1-v^2}}\psi' \;=0$$

Not Lorentz covariant! The easiest generalization is for ψ to have 4 components. Then we can have a Lorentz covariant first order DE with a mass term.

Dirac notation

$(x^0, x^1) \equiv (t, x)$ $\partial_0 \equiv \frac{\partial}{\partial x^0} \equiv \frac{\partial}{\partial t}$ $\partial_1 \equiv \frac{\partial}{\partial x^1} \equiv \frac{\partial}{\partial x}$

$\gamma^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\gamma^1 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Then rewrite $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial \psi}{\partial t} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial \psi}{\partial x} = 0$ as $\gamma^0 \partial_0 \psi + \gamma^1 \partial_1 \psi = 0$ or $\gamma^\mu \partial_\mu \psi = 0$

or even more elegantly as $\not{\partial} \psi = 0$

If you want to add a mass term, and to extend to 3 special dimensions, you need to have 4 components of ψ, the γ matrices become 4 x 4, you have 4 coordinates instead of 2, and there are 4 γ matrices instead of 2.

$$\not{\partial} \psi + im\psi = 0$$