Exercise on how to find new particles through scattering

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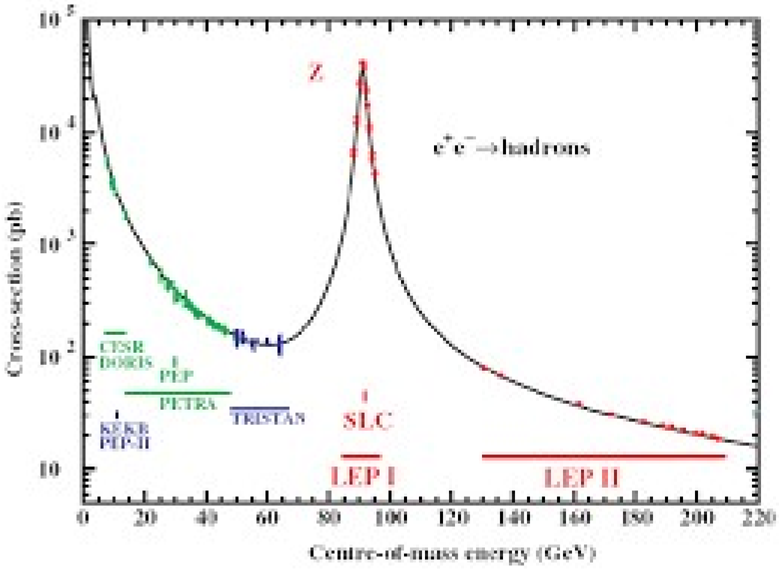
In the notes on scattering, I concluded with the example of a collision between an electron and a positron. I asserted that the modulus-squared of the S-matrix at very high energies becomes

$$|\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle|^2 \propto \frac{E^2}{[(4E^2 - M_Z^2)]^2} g(\theta, ...) \tag{1}$$

and then pointed out that this blows up when the centre-of-mass energy $2E$ is equal to $M_Z = 91$ GeV. An actual experimental graph shows a cross-section peak at that energy, and this is one of the ways that the Z particle was experimentally discovered.



In this exercise, we will "motivate" this S-matrix result from a

Green function which I will give you.

First, recall the LSZ theorem.

$$\begin{aligned}
&\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = \\
&\quad(-(p_1^{out})^2+m^2)(-(p_2^{out})^2+m^2)(-(p_1^{in})^2+m^2)(-(p_2^{in})^2+m^2) \\
&\quad[i\int d^4x_1 e^{-ip_1^{in}\cdot x_1}][i\int d^4x_2 e^{-ip_2^{in}\cdot x_2}][i\int d^4x_3 e^{+ip_1^{out}\cdot x_3}][i\int d^4x_4 e^{+ip_2^{out}\cdot x_4}] \\
&\quad\mathcal{G}(x_1,x_2,x_3,x_4)
\end{aligned} \tag{2}$$

- Remember that we define the Fourier transform of a function $f(x)$ as

  $$\tilde{f}(k) = \int dx e^{-ikx} f(x) \tag{3}$$

  where x and p are one-dimensional variables.

  Also remember the inverse transform theorem that

  $$f(x) = \frac{1}{2\pi}\int dx e^{ikx}\tilde{f}(k) \tag{4}$$

  Let $\tilde{\mathcal{G}}(k_1,k_2,k_3,k_4)$ be defined as the Fourier transform, with respect to each of the 4 arguments, of the Green function

  Prove that the RHS of equation (2) is

  $$\begin{aligned}
  &(-(p_1^{out})^2+m^2)(-(p_2^{out})^2+m^2)(-(p_1^{in})^2+m^2)(-(p_2^{in})^2+m^2) \\
  &\quad\frac{1}{(2\pi)^4}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-ip_1^{in}\cdot x_1 - ip_2^{in}\cdot x_2 + ip_1^{out}\cdot x_3 + ip_2^{out}\cdot x_4} \\
  &\quad\int d^4k_1 d^4k_2 d^4k_3 d^4k_4 e^{ik_1\cdot x_1 + ik_2\cdot x_2 + ik_3\cdot x_3 + ik_4\cdot x_4}\tilde{\mathcal{G}}(k_1,k_2,k_3,k_4)
  \end{aligned} \tag{5}$$

- **Then reorganize terms to obtain**

  $$\begin{aligned}
  &((p_1^{out})^2-m^2)((p_2^{out})^2-m^2)((p_1^{in})^2-m^2)((p_2^{in})^2-m^2) \\
  &\quad\frac{1}{(2\pi)^4}\int d^4k_1 d^4k_2 d^4k_3 d^4k_4 d^4x_1 d^4x_2 d^4x_3 d^4x_4 \\
  &\quad e^{i(-p_1^{in}+k_1)\cdot x_1 + i(-p_2^{in}+k_2)\cdot x_2 + i(p_1^{out}+k_3)\cdot x_3 + i(p_2^{out}+k_4)\cdot x_4}\tilde{\mathcal{G}}(k_1,k_2,k_3,k_4)
  \end{aligned} \tag{6}$$

- . We're almost there. Recall that

$$\frac{1}{2\pi}\int dx e^{ikx} = \delta(k)$$

Do all of the x_i integrals in expression (6), use the above δ function identity and then integrate over the k_i to transform expression (6) to

$$((p_1^{out})^2 - m^2)((p_2^{out})^2 - m^2)((p_1^{in})^2 - m^2)((p_2^{in})^2 - m^2)\tilde{\mathcal{G}}(p_1^{in}, p_2^{in}, -p_1^{out}, -p_2^{out}) \tag{7}$$

- Recall that we started with the RHS of equation (2) and ended up with (7). So we've shown that the S-matrix element is

$$\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = \\ ((p_1^{out})^2 - m^2)((p_2^{out})^2 - m^2)((p_1^{in})^2 - m^2)((p_2^{in})^2 - m^2)\tilde{\mathcal{G}}(p_1^{in}, p_2^{in}, -p_1^{out}, -p_2^{out}) \tag{8}$$

This is a general result independent of the form of the Green function. I promised to give you an approximation to the Green function (what is called the lowest-order term in the asymptotic expansion) for electron-positron scattering. To make things a bit simpler, I'm going to pretend that all particles are scalars, and that the incoming and outgoing particles have the same mass. That's the assumption that went into the particular form of the LSZ theorem that we've been using. If the scattered particles are associated with the field ϕ, if the Z-particle is associated with the field $\mathcal{Z}$, and if the Lagrangian of the theory contains an interaction term of the form $\lambda\phi^2(x)\mathcal{Z}(x)$, then the approximate Green function (in Fourier space)will be

$$\tilde{\mathcal{G}}(p_1^{in}, p_2^{in}, -p_1^{out}, -p_2^{out}) = \\ K\tilde{\Delta}_m(p_1^{in})\tilde{\Delta}_m(p_2^{in})\tilde{\Delta}_m(-p_1^{out})\tilde{\Delta}_m(-p_2^{out})\tilde{\Delta}_{M_Z}(p_1^{in} + p_2^{in}) \tag{9}$$

where $\tilde{\Delta}_M(k) \equiv \frac{1}{k^2 - M^2}$, K is a constant, m is the mass of the ϕ particles and M_Z is the mass of the Z particle.

Using equation (9), plug into equation (8) to obtain

$$\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = \tilde{\Delta}_{M_Z}(p_1^{in} + p_2^{in}) = \frac{K}{(p_1^{in} + p_2^{in})^2 - M_Z^2} \tag{10}$$

Notice that lots of stuff cancelled.

- Recall from my notes on scattering, that

$$
\begin{aligned}
p_1^{in} &= (E, \vec{p}) \\
p_2^{in} &= (E, -\vec{p})
\end{aligned}
\tag{11}
$$

Use this to obtain from equation (10) that

$$\langle p_1^{out}, p_2^{out} | S | p_1^{in}, p_2^{in} \rangle = \frac{K}{(2E)^2 - M_Z^2}$$

so

$$|\langle p_1^{out}, p_2^{out} | S | p_1^{in}, p_2^{in} \rangle|^2 = \frac{|K|^2}{[(2E)^2 - M_Z^2]^2}$$

This has the same denominator as in equation (1).