Introduction to Perturbation Theory

Bill Celmaster

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Start with a toy field theory with a potential energy term $V(\phi) = \lambda\phi^3$. WHAT IS WRONG WITH THIS THEORY?

$$S[\lambda, J, \phi] = \int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi - \lambda\phi^3) + i\epsilon \tag{1}$$

In all expressions below, we are implicitly taking $\lim_{e\to 0^+}$.

1 General theory of perturbative expansions of non-free theories

This section follows section 4.1 in Kachelriess and Zee section 1.7 (especially around page 48 entitled "Perturbative Field Theory") but I use a cubic interaction term $\lambda\phi^3$ and they focus on a quartic interaction $\lambda\phi^4$. WHY?

- We are interested in the Green functions

$$\begin{aligned} G_\lambda(x_1, ..., x_n) &= \frac{1}{Z[\lambda, 0]}\int \mathcal{D}\phi\phi(x_1)...\phi(x_n)e^{iS[\lambda,J,\phi]} \\ &= (-i)^n\frac{1}{Z[\lambda, 0]}\frac{\delta^n Z[\lambda, J]}{\delta J(x_1)...\delta J(x_n)}|_{J(x)=0} \end{aligned} \tag{2}$$

  where

$$Z[\lambda, J] = \int \mathcal{D}\phi e^{iS[\lambda,J,\phi]} \tag{3}$$

  In what follows, I will sometimes substitute notation: $\frac{\delta}{\delta J(x_1)}$ will become $\frac{\delta}{\delta J_{x_1}}$ and so on. The two expressions mean the same thing. The second expression is meant to remind you of more familiar notation like $\frac{\partial}{\partial J_i}$. See the Appendix for more information.

- We can compute this analytically when $\lambda = 0$.

$$Z[0,J] = e^{-\frac{1}{2}i\int d^4y d^4y' J(y)\Delta_F(y-y')J(y')} \tag{4}$$

where

$$\Delta_F(x-x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k^2-m^2+i\epsilon} \tag{5}$$

- Recall equation (2) and expand the action $S[\lambda, J, \phi]$ from equation (1)

$$\begin{aligned}
G_\lambda(x_1,...,x_n) &= \frac{1}{Z[\lambda,0]}\int \mathcal{D}\Phi\phi(x_1)...\phi(x_n)e^{i\int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2+J\phi-\lambda\phi^3)+i\epsilon} \\
&= \frac{1}{Z[\lambda,0]}\int \mathcal{D}\phi\phi(x_1)...\phi(x_n)e^{i\int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2)+i\epsilon}e^{-i\int d^4x\lambda\phi^3} \\
&= \frac{1}{Z[\lambda,0]}\int \mathcal{D}\phi\phi(x_1)...\phi(x_n)e^{iS[0,J,\phi]}e^{-i\lambda\int d^4x\phi^3}
\end{aligned} \tag{6}$$

- Suppose λ is small, so that we can Taylor-expand the term in red. The Taylor expansion involves products of ϕ's with the exponential of a free ($\lambda = 0$) action. All of these are Green functions G_0 and can be computed analytically

$$\begin{aligned}
&G_\lambda(x_1,...,x_n) \\
&= \frac{1}{Z[\lambda,0]}\int \mathcal{D}\phi\phi(x_1)...\phi(x_n)e^{iS[0,J,\phi]}(1-i\lambda\int d^4x\phi(x)^3-\frac{\lambda^2}{2}\int d^4x\phi(x)^3\int d^4x'\phi(x')^3+...) \\
&= G_0(x_1,...,x_n)-i\lambda\int d^4xG_0(x,x,x,x_1,...,x_n) \\
&\quad -\frac{\lambda^2}{2}\int d^4xd^4x'G_0(x,x,x,x',x',x',x_1,...,x_n)+...
\end{aligned} \tag{7}$$

2 Computing J derivatives of Z[0,J]

Recall the expression for Z[0,J] given in equation (4).

$$Z[0,J] = e^{-\frac{1}{2}i\int d^4y d^4y' J(y)\Delta_F(y-y')J(y')}$$

All Green functions are obtained by taking derivatives with respect to J **and then setting** $J = 0$.

- One derivative:

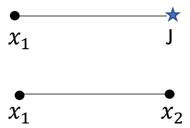
$$\frac{\delta}{\delta J(x_1)}(Z[0,J]) = Z[0,J][-i\int d^4y J(y)\Delta_F(y-x_1)] \tag{8}$$

- Two derivatives: Apply one derivative to above.

$$\frac{\delta^2}{\delta J(x_2)\delta J(x_1)}(Z[0,J]) = \frac{\delta}{\delta J(x_2)}(Z[0,J][-i\int d^4y J(y)\Delta_F(y-x_1)])$$
$$= Z[0,J][-i\Delta_F(x_2-x_1) + [-i\int d^4y J(y)\Delta_F(y-x_1)][-i\int d^4y J(y')\Delta_F(y'-x_2)]] \tag{9}$$

- More generally, it's easiest to use diagrams.

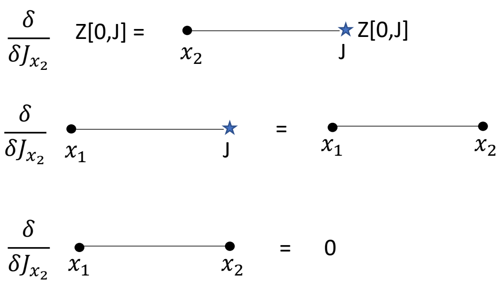
3 Diagrams



$$-i \int d^4y\, J(y) \Delta_F(y - x_1)$$

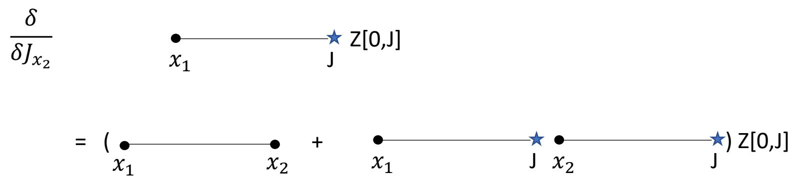
$$-i\, \Delta_F(x_2 - x_1)$$

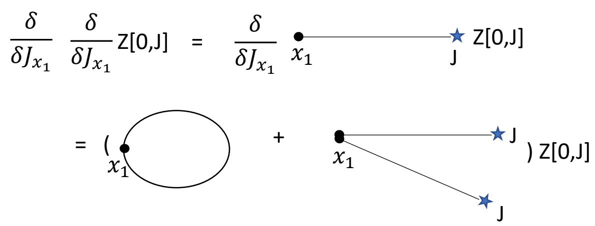
DIAGRAM RULES



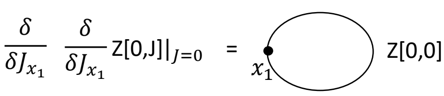
When we set J=0, all lines with 'J' are set to 0, and Z is set to Z[0,0]

Examples

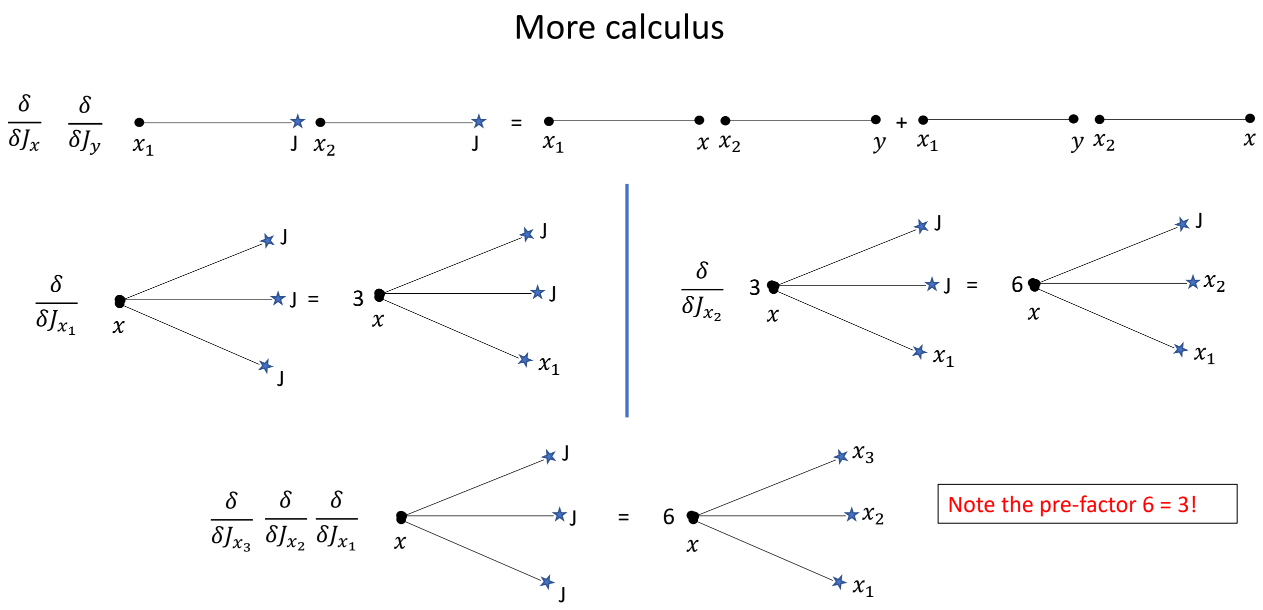




Notice what happens when both J indices are the same



When we set J=0 only the bubble and Z[0,0] are left



4 Appendix: Discrete version

What follows will parallel Zee's "child problem" (see section 1.7 subsection "Propagation: from here to there").

The notation $\frac{\delta}{\delta J_{x_1}}$ or alternatively $\frac{\delta}{\delta J(x_1)}$ may be slightly unfamiliar. If you find it confusing, it might help to think about all continuous variables as being discrete. It also helps to stay in 1 dimension, rather than 4.

- x_1 becomes 1D rather than 4D
- x_1 becomes an integer i_1, etc.
- integrals over d^4x_i become sums over i_1, etc.
- $J(x)$ becomes the vector $J(i)$ sometimes written J_i.
- $\frac{\delta}{\delta J_{x_i}}$ or $\frac{\delta}{\delta J(x_i)}$ becomes $\frac{\partial}{\partial J_i}$
- $\Delta_F(x-y)$ becomes the matrix $\mathcal{D}_{ij}$
- $Z[\lambda, J]$ becomes $Z_d[\lambda, J]$

With these substitutions,

$$Z[0, J] = e^{-\frac{1}{2}i\int d^4y d^4y' J(y)\Delta_F(y-y')J(y')}$$

becomes

$$Z_d[0, J] = e^{-\frac{1}{2}i\sum_{ij} J_i J_j \mathcal{D}_{ij}}$$

The first J derivative substitutes the expression

$$\frac{\delta}{\delta J(x_1)}(Z[0, J]) = [-i\int d^4y J(y)\Delta_F(y - x_1)]Z[0, J]$$

to become

$$\frac{\partial}{\partial J_{i_1}}(Z_d[0, J]) = [-i\sum_k J_k \mathcal{D}_{ki_1}]Z_d[0, J]$$

The expressions (in red) using discrete indices, should feel somewhat more familiar than the expressions using continuous indices, so if you are ever unsure about the calculus of continuous indices, make the above substitutions and check results with discrete indices.