Kachelriess pp 33-36

Bill Celmaster

August 18, 2020

1 General Outline

Kachelriess pages 33-36 cover the action principle for quantum fields. This treatment generalizes Kachelriess section 1.1, and also generalizes (end of section 3.1) the path integral of equation 2.40.

Below, I'll give references to this material as covered by other authors.

I'll also provide a few (related) exercises that I concocted (I couldn't find anything that grabbed me in any of the other references.)

Finally, I'll make some comments that might help with the section and with the exercise.

WARNING. Be careful if you read ahead on page 37. I think Kachelriess 'mis-speaks' after equation 3.18. I, for one, am confused by what he's trying to say and believe it can be said differently (correctly?)

2 Other references

Every text gets to this material in a different way. I'll give you the primary page references but you may need to read other parts of those texts in order to understand what the author is saying.

- **Lancaster**: Sections 5.3 and part of 5.4 pages 54-56. Sections 7.1 and 7.2 pages 64-66.
- **Klauber**: pages 48-50 (this may be useful for anyone wanting to see equations written out)
- **Schwartz**: Sections 3.1 and 3.2 pages 29-32. I plan to make copies of those pages for those of you who don't have the text

3 Exercises

The intention of these exercises is to have you follow Kachelriess but applying some of the generalities to a specific Lagrangian. I've tried to make the exercises independent of one another so that if you don't solve one , you can still do the others.

For the sake of simplicity, assume we have only one spacial dimension. So the world is, altogether, 2-dimensional (one time and one space dimension).

Let

$$\mathcal{L}(\Phi, \partial_\mu \Phi) = \frac{1}{2}[(\partial^0\Phi)^2 - (\partial^1\Phi)^2) + \gamma\Phi^2 - \lambda\Phi^4]$$

where Φ is a function of t and x which we will write as x^0 and x^1, i.e. $\Phi(x^0, x^1)$. Let

$$S[\Phi] = \int dx^0 dx^1 \mathcal{L}(\Phi, \partial_\mu \Phi)(x^0, x^1)$$

where x^0 and x^1 range from $-\infty$ to ∞ and furthermore, $\Phi(x^0, x^1) \to 0$ as $(x^0, x^1) \to (\pm\infty, \pm\infty)$ (those boundary conditions will be important when integrating by parts).

1. Show that the action is Lorentz-invariant. (This is meant to give you a bit of insight into what Kachelriess is saying in the first two sentences of section 3.1.) What this means is "show that the action doesn't change when space and time are transformed in the usual way, and when the field transforms as a scalar (see below)." In general, we will only be interested in Lorentz-invariant Lagrangians, so that will constrain which Lagrangians are possible. Steps:

   - Write down the transformation equations (hint – see my attached presentation from April on the Dirac equation – the way Φ transforms in those notes is what is meant by "transforms as a scalar").
   - The goal is to show that $S[\Phi] = S'[\Phi']$ where $S'[\Phi'] = \int dx'^0 dx'^1 \mathcal{L}'(\Phi', \partial_\mu \Phi')(x'^0, x'^1)$ and primed variables represent the Lorentz-transformed variables (see previous item).
     - Start with $S[\Phi]$ and, using the transformation equations, rewrite the integrand in terms of primed variables (hint – see the April presentation). The integrand should end up looking like the integrand of $S'[\Phi']$.
     - However, since we started with $S[\Phi]$, the integration variables are still $dx^0 dx^1$. Transform these to $dx'^0 dx'^1$ (hint – calculate the Jacobian determinant for the transformation matrix)

2. Find the extremum of S, i.e., find Φ so that $\delta S[\Phi] = 0$. You can do this directly following the steps in Kachelriess 3.1 to 3.3, but with the specific Lagrangian for this exercise. Or you can go straight to the Euler-Lagrange equation 3.4 and apply it to this Lagrangian. The first approach might be more instructive. The second is probably easier.

3. - Following equations 3.12 through 3.14, find the canonical momentum (aka 'canonically conjugated momentum') for our Lagrangian, and the Hamiltonian.
   - Let $\gamma < 0$ and $\lambda = 0$. Rewrite γ in terms of a variable m so that our Lagrangian is equivalent (in 2D) to equation 3.7. What is the minimum value of the Hamiltonian?
   - What if $\gamma > 0$? What is the minimum value of the Hamiltonian? Comment on this.
   - Find combinations of γ and λ so that
     - The Hamiltonian is bounded from below (i.e., the minimum is not $-\infty$) and the minimum is $\Phi(x^0, x^1) = 0$.
     - The Hamiltonian is bounded from below (i.e., the minimum is not $-\infty$) but the minimum is **not** $\Phi(x^0, x^1) = 0$. Sketch an example graph of the potential V.

4 Some comments on Kachelriess

- What matters in Section 3.1 are equations 3.1, 3.4 and 3.5. These generalize respectively 1.1, 1.4 and 2.40. I'm not crazy about the discussions surrounding these, since they tend to make references to things that haven't been studied. The derivations from 3.1 through 3.4 are fine but terse. It may help to look at chapter 1 or other references, and if you can do the exercises, that might also make things less general and easier to follow.

- Pages 34 through 36 are an attempt to justify the selection of the Lagrangian for the free scalar theory (equation 3.7). Maybe you'll like this treatment. I don't. There are interesting remarks in this section but as usual, they strike me as terse with references to ideas that aren't fully developed. At the bottom of page 36 is a brief discussion of 'expanding the potential around a minimum'. As far as I can tell, there's no reason given for why such an expansion should be made. At some point in the future it will make sense to explain that, but not yet. Furthermore, at some point, it will make sense to explain why we should care (the answer is "spontaneous symmetry breaking") but not yet.

- Still, there's a couple of things in these pages that seem worth elaborating and which are a partial focus of the exercises.
  - Lagrangians are chosen so that the action is Lorentz-invariant. The free scalar Lagrangian is a particularly simple-looking example. Kachelriess's approach is indirect. He argues that the Klein-Gordon equation is 'naturally' Lorentz-invariant and then works backward from a Euler-Lagrange equation (which looks like the Klein-Gordon equation) to argue that this comes from the free scalar Lagrangian. In the first exercise, I suggest that you show directly that the action is invariant. That seems to me to be more straightforward.
  - Various times, Kachelriess tells us that energies better be positive – or more precisely, bounded from below – otherwise bad things happen. The physics literature abounds with such statements but rarely goes into satisfying explanations of why this is a requirement. Still, it seems intuitively reasonable as I'll now indicate. In classical mechanics, consider the potential energy between a positive and negative charge, $V(r) = -\frac{k}{r}$. If you start with the two charges at rest at distance $r = R$ from one another, then by the time they 'collide' at $r = 0$ they will be traveling infinitely fast and will therefore have infinite kinetic energy. This situation is considered to be unacceptable, but also unphysical. We fix this problem by noting that the charges must have a non-zero radius and that once the two particles are closer than that radius, the potential energy drops off. In other words, the actual potential energy is modified so that it isn't (negatively) infinite at $r = 0$. By contrast, suppose the two charges had the same sign. Then they would repel and the velocity would eventually stabilize when the charges are sufficiently far apart. In that case there wouldn't be infinite kinetic energy. This all leads to the intuition that a physically plausible potential is one which must be bounded from below.