Kachelriess pp 37-40 v1.1

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1 General Outline

Kachelriess pages 37-40 cover the generating function for Green functions of quantum field theory. This treatment generalizes Kachelriess sections 2.3 and 2.4.

Below, I'll give references to this material as covered by other authors.

I'll also provide a few (related) exercises that I concocted (I couldn't find anything that grabbed me in any of the other references.)

Finally, I'll make some comments that might help with the section and with the exercise.

2 Other references

- **Lancaster**: Chapters 22-25 have significant overlap with Kachelriess sections 3.1 and 3.2. The Lancaster chapters are considerably clearer and considerably more thorough. However, some of the treatment requires material in earlier chapters of Lancaster and which are entirely missing from Kachelriess (at least prior to Chapter 3). Also, you'll find there's material in Kachelriess Chapter 3 which isn't covered in Lancasters chapters 22-25.
- **Schwartz**: Section 6.2 derives the Feynman propagator, but from a completely different perspective than what was done with the path integral. Section 14.3 discusses generating functionals in scalar field theory and overlaps with Kachelriess on pages 261 and 262.

3 Exercises

Since so much of this section involves Fourier transforms, it might be helpful to get comfortable with various Fourier transform manipulations.

Start with a few 1-D identities. Consider a function $f(x)$. Define the Fourier transform $\tilde{f}(k) = \int dx e^{ikx} f(x)$. Then it can be proven that $f(x) = \int \frac{dk}{2\pi} e^{-ikx} \tilde{f}(k)$. I'll also use the notation $\mathcal{F}[f](k) \equiv \tilde{f}(k)$.

1. The following identities are commonly used, and you should get used to them. Prove them.

   - $\delta(x) = \int \frac{dk}{2\pi} e^{\pm ikx}$. (Hint: Note that for any function $h(x)$, $\int h(x)\delta(x) = h(0)$.)
   - Let f' be the function defined as $f'(x) = \frac{df}{dx}$. Then $\mathcal{F}[f'](k) = -ik\tilde{f}(k)$.
   - Let g be the function defined as $g(x) = \sum_{n=1}^{N} a_n \frac{d^n f(x)}{dx^n}$. Then $\mathcal{F}[g](k) = \sum_{n=1}^{N} a_n (-ik)^n \tilde{f}(k)$.

2. Let $\frac{d^2 f(x)}{dx^2} + m^2 f(x) = K(x)$. Find $\tilde{f}(k)$ in terms of $\tilde{K}(k)$.

3. Let $J(x)$ and $D(x)$ be two functions with Fourier transforms $\tilde{J}(k)$ and $\tilde{D}(k)$. Then define $\mathcal{I}[J] = \int dx dx' J(x) D(x - x') J(x')$. Show that $\mathcal{I}[J] = \int \frac{dk}{2\pi} \tilde{J}^*(k) \tilde{D}(k) \tilde{J}(k)$ where $\tilde{J}^*$ is the conjugate of $\tilde{J}$.

Next consider the 2-D Minkowski space with dot product defined by $a \cdot b = a_0 b_0 - a_1 b_1$ for any 2D vectors a and b. Let x and k be 2D vectors. The 2D Fourier transform is given by $\tilde{f}(k) = \int d^2x e^{ik\cdot x} f(x)$. It can be proven that $f(x) = \int \frac{d^2k}{(2\pi)^2} e^{-ik\cdot x} \tilde{f}(k)$.

1. Generalize the identities of exercise 1 to 2D. (You needn't prove them if you feel comfortable simply stating them.)

2. Consider the function $g(x) = \partial_0^2 f(x) - \partial_1^2 f(x) + m^2 f(x)$. Find $\tilde{g}(k)$ in terms of $\tilde{f}(k)$. Suppose for all values of k that $\tilde{f}(k) \neq 0$. Find the values of k so that $\tilde{g}(k) = 0$ and express those values by giving k_0 in terms of k_1 and m. These are the so-called mass-shell values of k.

4 Some comments on Kachelriess

- There's lots in this section. I'll outline the main points here. As usual, Kachelriess makes a bunch of passing remarks about lots of topics, and although those remarks are potentially interesting, they are inscrutable. Fortunately, they can be skipped.
  - $Z[J]$ is defined in 3.16 as the path integral for a free massive scalar theory with the addition of a linear source term analogous to 2.53.
  - $Z[J]$ is derived generically for a path integral of the form

    $$\mathcal{N} \int \mathcal{D}\Phi e^{i(\int d^4x d^4x' \frac{\Phi(x)A(x,x')\Phi(x')}{2} + \delta(x-x')J(x)\Phi(x))}$$

    and is given in 3.20 analogous to 2.70 (what Matthew's notes on section 2.4 calls *miraculous*). The expression involves $A^{-1}(x, x')$
  - For convenience, $W[J]$ is defined as $-i \log Z[J]$ in 3.21
  - Pages 38 - 40, in the section called "Propagator", analyzes A^{-1} for the case that A represents the free massive scalar theory. In that case, we define $\Delta(x, x') \equiv A^{-1}(x, x')$. This inverse is only well-defined when the Lagrangian contains an extra infinitesimal term.
    * After some manipulations, Kachelriess arrives at an expression for Δ_F, which is the value of A^{-1} when A represents a Lagrangian which adds to the mass an infinitesimal term whose effect is to replace m^2 by $m^2 - i\epsilon$. $\Delta_F(x - x')$ is given in 3.25a and **is the key result of this section.**
      · **WARNING!**. Kachelriess plays fast and loose with his use of Δ and Δ_F and their arguments. At different times he writes $\Delta_F(x - x')$, $\Delta(x, x')$ and $\Delta_F(k)$. This is sloppy. I am often guilty of similar sloppiness. First, notice that the (x, x') dependence for the propagator, is of the form $x - x'$. Secondly, notice that sometimes we take the Fourier transform of $\Delta_F(x - x')$ and this really should be written differently – for example $\tilde{\Delta}_F(k)$.
    * Although $\Delta_F(x - x')$ is an integral over all values of the 4-vector k, it turns out that the only values that contribute in the limit $\epsilon \to 0$ are on hyperplanes where $k_0 = \pm\sqrt{\vec{k} \cdot \vec{k} + m^2}$. This is derived finally in equation 3.28. Those hyperplanes are known as *the mass shell*.

* Although Kachelriess obtains a nice concise expression in 3.32 for $W[J]$, this expression is rarely used. Best to stick with 3.30, using $\Delta_F(x - x')$ in 3.25a

- The expression $\langle 0 + | 0 - \rangle_J$ appears in 3.16. If this doesn't bother you, then don't worry about it. On the other hand, if you are wondering exactly what that means, the good news is you really don't have to know. All that matters is that the right hand side of 3.16 is related to the scattering matrix. However, in case you're curious, I think we can regard $|0-\rangle_J$ as the vacuum state of $H_J(t)$ when $t \to -\infty$ where H_J is the Hamiltonian of the theory which includes the source term $J(x)\Phi(x)$. A similar interpretation applies to $\langle 0 + |_J$. Since the source is explicitly time-dependent, it breaks what would otherwise be time-translation invariance of the Lagrangian. Therefore (for reasons having to do with Noether's theorem and thus not obvious) the Hamiltonian depends on time (without the source term, the Hamiltonian is time-independent – hence conservation of energy). In particular, the vacuum states are different at $t = \pm\infty$. When there is no source term, there is only one vacuum (at least, that's true for "simple" theories).