Fourier Transforms – basic facts – Bill Celmaster September 2020

- FT

$$\mathcal{F}[f](k) \equiv \tilde{f}(k) = \int dx e^{ikx} f(x)$$

- Inverse FT

$$f(x) = \int \frac{dk}{2\pi} e^{-ikx} \tilde{f}(k)$$

- Delta function

$$\delta(x) = \int \frac{dk}{2\pi} e^{\pm ikx}$$

so, for example

$$\delta(x - x_1) = \int \frac{dk}{2\pi} e^{-ikx} e^{ikx_1}$$

thus

$$\tilde{F}[\delta(x - x_1)] = e^{ikx_1}$$

- Derivatives

$$\mathcal{F}[\frac{df}{}](k) = -ik\tilde{f}(k)$$

Using FT to solve PDE continued

- Let $\partial_\mu \partial^\mu f(x) + m^2 f(x) = J(x)$
- Then in Fourier space

$$\tilde{J}(k) = (-k_0^2 + k_1^2 + k_2^2 + k_3^2 + m^2)\tilde{f}(k)$$

.

- Solve in Fourier space

$$\tilde{f}(k) = -\frac{\tilde{J}(k)}{k \cdot k - m^2}$$

Our first bit of physics – using W[J] to find the potential energy of a charge

- In the Appendix, we "derive"

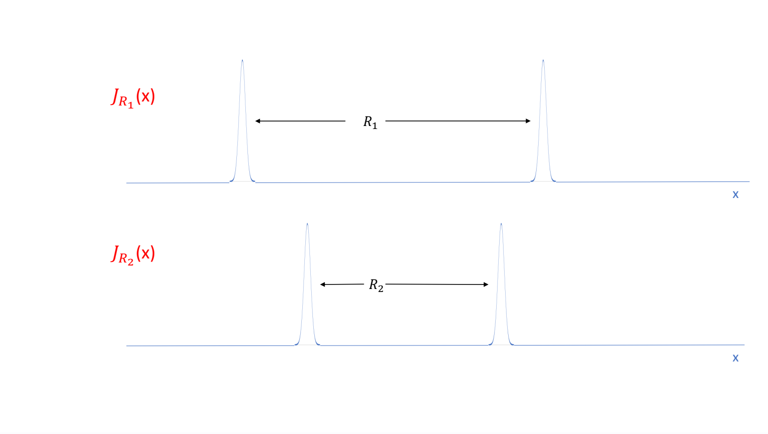
$$W[J] = -V(J)\tau$$

  when $J(t, \vec{x})$ is a function that can be written as $\Theta_H(t)\Theta_H(\tau - t)\hat{J}(\vec{x})$. In other words, $J(t, \vec{x})$ is 0 except in the interval of time between 0 and τ. During that interval, J is time-independent. Take τ large.
- Now use W to compute the potential V for the case of a massless scalar field (Coulomb potential) and a massive scalar field (Yukawa potential).

$W[J]$ for a source of separated "charges"

$\hat{J}(\vec{x}) = \delta(\vec{x} - \vec{x}_1) + \delta(\vec{x} - \vec{x}_2)$. Then vary the distance between x_1 and x_2 to find the energy's dependence on separation.

Figure: A source with two charges



Compute W[J]

- From the delta-function identity

$$\tilde{\tilde{J}}(k) = (e^{i\vec{k}\cdot\vec{x}_1} + e^{i\vec{k}\cdot\vec{x}_2}) \int_0^\tau dt e^{-ik_0 t}$$

- Start with Kachelriess equation 3.32

$$W[J] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J(k)^* \frac{1}{k^2 - m^2 + i\epsilon} J(k)$$

$$= -\frac{1}{2}\int_0^\tau dt \int_0^\tau dt' \int \frac{dk_0}{2\pi} e^{ik_0 t} e^{-ik_0 t'} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(x_1 - x_2)}}{k^2 - m^2 + i\epsilon} + \ldots$$

$$= -\frac{1}{2}\int_0^\tau dt \int dk_0 e^{ik_0 t} \int_0^\tau \frac{dt'}{2\pi} e^{-ik_0 t'} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(x_1 - x_2)}}{k^2 - m^2 + i\epsilon} + \ldots$$

$$\approx -\frac{1}{2}\tau \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(x_1 - x_2)}}{-\vec{k}\cdot\vec{k} - m^2 + i\epsilon} + [(x1, x2) \to (x2, x1)] + [(x1, x2) \to (x1, x1)] + [(x1, x2) \to (x2, x2)]$$

where the sequence in red has $\approx \delta(k_0)$ that sets k_0 to ≈ 0 and the ellipsis is explained in the last line

Computing the potential

Putting everything together, we have

$$-V(J)\tau = W[J] \approx -\frac{1}{2}\tau \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(x_1-x_2)}}{-\vec{k}\cdot\vec{k} - m^2 + i\epsilon} + \dots$$

so

$$\begin{aligned} V(J) &\approx \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(x_1-x_2)}}{-\vec{k}\cdot\vec{k} - m^2 + i\epsilon} + \dots \\ &= -\frac{1}{2}[\frac{e^{-m|\vec{x}_1-\vec{x}_2|}}{4\pi|\vec{x}_1-\vec{x}_2|}] + -\frac{1}{2}[\frac{e^{-m|\vec{x}_2-\vec{x}_1|}}{4\pi|\vec{x}_2-\vec{x}_1|}] \dots \\ &= -\frac{e^{-m|\vec{x}_1-\vec{x}_2|}}{4\pi|\vec{x}_1-\vec{x}_2|} + \text{self-energy terms } ((x_1, x_1) \text{ and } (x_2, x_2) \text{ terms}) \end{aligned}$$

The self-energy terms are separation-independent so can be treated as a constant to be subtracted.

What does it all mean?

- Let $m \to 0$. The potential energy dependence is the Coulomb potential
$$V_C(r) = E_0(J) = -\frac{1}{4\pi r}$$
- When $m \neq 0$ we have the Yukawa potential
$$V_Y(r) = E_0(J) = -\frac{e^{-mr}}{4\pi r}$$
- Yukawa matched this potential to the observed scale of nuclear interactions, and predicted in the 1930's that $m \approx 100$ MeV.
- Yukawa realized from field theory that m represents a particle mass. The pion, with mass 140 MeV, was discovered in 1947.

Appendix – relate $W[J]$ to a potential

Review.

1. Assert – **Everything** in nature can be inferred from the scattering matrix (how particles 'bounce off each other')
2. The scattering matrix is easily computed from $G(x_1, ..., x_n)$.
3. (Generalizing 2.55 or see section 3.3)

$$G(x_1, ..., x_n) = \mathcal{N} \int \mathcal{D}\Phi\Phi(x_1)...\Phi(x_n) exp^{i\int d^4x\mathcal{L}(x)}$$
$$= (-i)^n \frac{1}{Z[0]} \frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)}|_{J(x)=0}$$

   where

4. $Z[J]$ is the path integral for $\mathcal{L}[J] \equiv \mathcal{L} + J(x)\Phi(x)$.
5. $$Z[J] = exp(iW[J])$$

So far, the only role for J is as a trick for computing Green functions.

Appendix – relate W[J] to a potential, cont'd

BUT ... we can ask the question "what kind of physical system would have a Lagrangian $\mathcal{L}[J] \equiv \mathcal{L} + J(x)\Phi(x)$?"

That insight comes from classical physics, especially from Maxwell's equations in the Lagrangian formalism.

$\mathcal{L}[J]$ is the Lagrangian for field theory where the field is coupled to an external charged source $J(x)$ (e.g. a heavy particle). That theory has a Hamiltonian $H[J]$.

Remember how we derived the path integral $\mathcal{Z}$. When the Hamiltonian is time-independent, Z is proportional to $\langle 0|e^{-iH\tau}|0\rangle$ where the path integration is taken over a time range τ.

- The vacuum state $|0\rangle$ is the state of lowest energy of H with $H|0\rangle = E_0|0\rangle$ so $Z = \mathcal{N}e^{-iE_0\tau}$.
- This is true (if J is time-independent) also of the J-dependent Hamiltonian $H[J]$ so $Z[J] = \mathcal{N}[_J\langle 0|e^{-iH[J]\tau}|0\rangle_J] = \mathcal{N}e^{-iE_0(J)\tau}$.

Appendix – relating W[J] to a potential, conclusion

- Recap: $E_0[J]$ is the lowest energy of a system consisting of a scalar field theory coupled to a time-independent charge distribution, J.
- The effect of J is to modify the system energy from what it would have been in the absence of a source. So interpret the energy-difference as the potential energy due to the source.
- $$Z[J] \equiv exp(iW[J]) = exp(-iE_0(J)\tau)$$

  so

  $$W[J] = -E_0(J)\tau \ = -V(J)$$