- Definition of Green function:

$$G(x_1, ...x_n) \equiv \langle 0| T\{\phi(x_1)...\phi(x_n)\}|0\rangle \tag{1}$$

where x_i is a 4-vector and time-ordering is with respect to the time component.

- Definition of the path integral:

$$e^{iW[J]} \equiv Z[J] \equiv \int \mathcal{D}\phi e^{i(S[\phi]+\int d^4xJ(x)\phi(x)+i\epsilon)} \tag{2}$$

- Deriving Green function from the path integral:

$$\begin{aligned} G(x_1, ..., x_n) &= \frac{q}{Z[0]} \int [\mathcal{D}\phi]\phi(x_1)...\phi(x_n)e^{i(S[\phi]+i\epsilon)} \\ &= (-i)^n \frac{1}{Z[0]} \frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)}|_{J(x)=0} \end{aligned} \tag{3}$$

Computing the Green's function on a super-duper computer

Simplify above to one dimension (time) and replace ϕ by q.

$$G(t_a, t_b) == \frac{\int (\prod_{t_i}^{\infty} dq_{t_i})(q_{t_a} q_{t_b} \prod_{t_j} e^{i(L(q(t_j), q(t_{j+1})) + i\epsilon)})}{\int (\prod_{t_i}^{\infty} dq_{t_i}) \prod_{t_j} e^{i(L(q(t_j), q(t_{j+1})) + i\epsilon)}}$$

The RHS is an infinite-dimensional integral. Actually, it isn't correct to label the integrals by a discrete index i, but we could set the whole thing up as a limit.

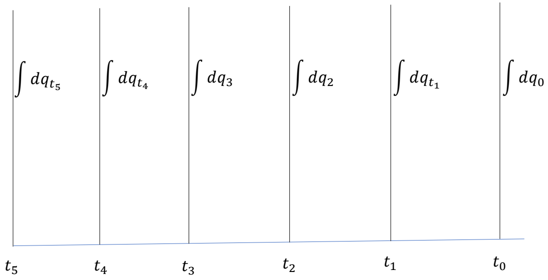
Approximate it numerically by picking a finite number of integrals, $\prod_{i=0}^{N} dq_{t_i}$ where t_i include t_a and t_b.

Figure: Numerical approximation to the path integral

Numerical approximation of path integral of F

Multi-dimensional integral of F, integrated (numerically) along each line.

$\int\int\int\int\int\int dq_{t_5}dq_{t_4}dq_{t_3}dq_{t_2}dq_{t_1}dq_{t_0}\, F(q_{t_5}, q_{t_4}, q_{t_3}\,, q_{t_2}, q_{t_1}, q_{t_0})$



$$\int dq_{t_5} F(q_{t_5}, \ldots) \approx \sum_i F(x_i, \ldots)$$

What to do without a supercomputer

The *lattice* (using supercomputers) approach is an approximation method only good for computing certain low-energy phenomena.

Other approximations can be done using the fact that $W[J]$ can be exactly computed when $S[\phi]$ is a 2nd degree polynomial in ϕ .

- The source method: Compute the effect of an external source on the field of interest (e.g. ϕ).
- Perturbation theory: Assume that terms in $e^{iS[\phi]}$ of higher order in ϕ can be Taylor-expanded and treated as a series of Green functions.

Example of $W[J]$ when $S[\phi]$ is a 2nd degree polynomial in ϕ:

- Let $S[\phi] = \int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi) + i\epsilon$
- Then $W[J] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J(k)^* \frac{1}{k^2 - m^2 + i\epsilon} J(k)$

Applying the source method to obtain the Yukawa potential

Last time, I started with the assumption that

$$W[J] = -V(J)T$$

. I picked a source J representing two charged particles and then derived an expression for $W[J]$. This led to the Yukawa potential.

Today let's show

$$W[J] = -V(J)T$$

.

So far, the only role for J is as a trick for computing Green functions.

BUT ... we can ask the question "what kind of physical system would have a Lagrangian $\mathcal{L}[J] \equiv \mathcal{L} + J(x)\Phi(x)$?"

How J changes the Hamiltonian

That insight comes from classical physics, especially from Maxwell's equations in the Lagrangian formalism.

$\mathcal{L}[J]$ is the Lagrangian for field theory where the field is coupled to an external charged source $J(x)$ (e.g. a heavy particle). That theory has a Hamiltonian $H[J]$.

Remember how we derived the path integral $\mathcal{Z}$. When the Hamiltonian is time-independent, Z is proportional to $\langle 0|e^{-iH\tau}|0\rangle$ where the path integration is taken over a time range τ.

- The vacuum state $|0\rangle$ is the state of lowest energy of H with $H|0\rangle = E_0|0\rangle$ so $Z = \mathcal{N}e^{-iE_0\tau}$.
- This is true (if J is time-independent) also of the J-dependent Hamiltonian $H[J]$ so $Z[J] = \mathcal{N}[{}_J\langle 0|e^{-iH[J]\tau}|0\rangle_J] = \mathcal{N}e^{-iE_0(J)\tau}$.

Relating W[J] to a potential

- Recap: $E_0[J]$ is the lowest energy of a system consisting of a scalar field theory coupled to a time-independent charge distribution, J.
- The effect of J is to modify the system energy from what it would have been in the absence of a source. So interpret the energy-difference as the potential energy due to the source.
- $$Z[J] \equiv exp(iW[J]) = exp(-iE_0(J)\tau)$$

  so

  $$W[J] = -E_0(J)\tau \; = -V(J)$$
- One important detail I skipped: The path integral represents vacuum to vacuum transitions ONLY when $i\epsilon$ is included AND the time interval ranges from $-\infty$ to $+\infty$. The proper way to deal with that is for the source J to be turned on at some point and to remain constant for a duration of time τ, at which point the source is turned off.

Annihilation and creation operators pp 44-46

- Most treatments start with annihilation and creation operators and then derive Green functions and eventually path integrals.
- Kachelriess has the challenge of starting with path integrals, then deriving Green functions and finally relating these to annihilation and creation operators and particles.
- This approach is indirect. Kachelriess hypothesizes a form for the field (equation 3.48) with the operators and Hilbert space specified by equation 3.49. This is the context where the states in the Hilbert space are associated with particles.
- He then shows that this hypothesis leads to the same propagator he obtained with the path integral.
- A certain amount of terminology is then introduced – notably *negative energy* and *virtual particles*. I don't find that terminology helpful but it's common. It has mostly to do with the exponent of $e^{\pm ik\cdot x}$ in various expressions. The positivity of energy has to do with the sign of the exponent and the virtual'ness has to do with the value of k^2.