Exercise 1

Lorentz invariance of
$\mathcal{S}(\Phi) = \frac{1}{2}\int dtdx[(\partial_t\Phi)^2 - (\partial_x\Phi)^2 + \beta\Phi^2 - \lambda\Phi^4](t,x)$

- Transformations:

$$t'(t,x) = \gamma(v)(t+vx)$$
$$x'(t,x) = \gamma(v)(vt+x)$$
$$\Phi(t,x) = \Phi'(t',x')$$

  where $\gamma(v) = \frac{1}{1-v^2}$.

- Transforming $\mathcal{L}(\Phi, \partial_\mu\Phi)$ – Chain rule:

$$\frac{\partial\Phi(t,x)}{\partial t} = \frac{\partial\Phi'(t'(t,x),x'(t,x))}{\partial t} = \frac{\partial\Phi'}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial\Phi'}{\partial t'}\frac{\partial t'}{\partial t} = \frac{\partial\Phi'}{\partial x'}v\gamma(v) + \frac{\partial\Phi'}{\partial t'}\gamma(v)$$

Exercise 1 transforming $\mathcal{L}$ continued

- Repeat chain rule:

$$\frac{\partial^2\Phi(t,x)}{\partial t^2} = 2\frac{\partial^2\Phi'(t',x')}{\partial t'\partial x'}(v\gamma^2(v)) + \frac{\partial^2\Phi'(t',x')}{\partial x'^2}(v^2\gamma^2(v)) + \frac{\partial^2\Phi'(t',x')}{\partial t'^2}(\gamma^2(v))$$

- Similarly:

$$-\frac{\partial^2\Phi(t,x)}{\partial x^2} = -2\frac{\partial^2\Phi'(t',x')}{\partial t'\partial x'}(v\gamma^2(v)) - \frac{\partial^2\Phi'(t',x')}{\partial x'^2}(\gamma^2(v)) - \frac{\partial^2\Phi'(t',x')}{\partial t'^2}(v^2\gamma^2(v))$$

Exercise 1 transforming $\mathcal{L}$ continued

- Add to get the kinetic term:

$$\frac{\partial^2\Phi(t,x)}{\partial t^2} - \frac{\partial^2\Phi(t,x)}{\partial x^2}$$
$$= \gamma^2(v)(1-v^2)\left(\frac{\partial^2\Phi'(t',x')}{\partial t'^2} - \frac{\partial^2\Phi'(t',x')}{\partial x'^2}\right)$$
$$= \frac{\partial^2\Phi'(t',x')}{\partial t'^2} - \frac{\partial^2\Phi'(t',x')}{\partial x'^2}$$

- Potential term

$$\beta\Phi^2(t,x) - \lambda\Phi^4(t,x) = \beta\Phi'^2(t',x') - \lambda\Phi'^4(t',x')$$

- Altogether

$$\mathcal{L}(\Phi,\partial_\mu\Phi)(t,x) = \mathcal{L}(\Phi',\partial_\mu\Phi')(t',x')$$

Exercise 1 transforming the metric

- The Jacobian is

$$\mathcal{J} = \begin{pmatrix} \frac{\partial t}{\partial t'} & \frac{\partial t}{\partial x'} \\ \frac{\partial x}{\partial t'} & \frac{\partial x}{\partial x'} \end{pmatrix} = \begin{pmatrix} \gamma(v) & -v\gamma(v) \\ -v\gamma(v) & \gamma(v) \end{pmatrix}$$

  where we had to rewrite the transformation equations for x and t in terms of x' and t'.
- The determinant of the Jacobian is

$$det\mathcal{J} = \gamma^2(v)(1 - v^2) = 1$$

- So

$$dtdx = |det\mathcal{J}|dt'dx' = dt'dx' \qquad (1)$$

- Both the metric and the Lagrangian are Lorentz invariant so this proves $S[\Phi] = S'[\Phi']$.

Exercise 2 using equation 3.4

- Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \Phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\right) \tag{2}$$

- Displaying terms with no derivatives of Φ

$$\mathcal{L}(\Phi, \partial_\mu \Phi) = \frac{1}{2}[\beta \Phi^2 - \lambda \Phi^4 + ...]$$

So $\frac{\partial \mathcal{L}}{\partial \Phi} = \beta \Phi - 2\lambda \Phi^3$

- Displaying terms with derivatives of Φ

$$\mathcal{L}(\Phi, \partial_\mu \Phi) = \frac{1}{2}[(\partial_t \Phi)^2 - (\partial_x \Phi)^2 + ...]$$

So $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\right) = \partial_t \partial_t \Phi - \partial_x \partial_x \Phi$

Exercise 2 continued

Putting both terms together, the Euler-Lagrange equation becomes

$$\Box\Phi = \beta\Phi - 2\lambda\Phi^3$$

or

$$(\Box - \beta)\Phi = -2\lambda\Phi^3$$

The left side should be familiar from Kachelriess equation 3.9. But the right side makes this equation difficult (impossible?) to solve in terms of known special functions.

Exercise 3 – the Hamiltonian

- Canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial_t \Phi} = \partial_t \Phi$$

similarly to what we had in Exercise 2.

- Hamiltonian

$$\begin{aligned}\mathcal{H} &= \pi \partial_t \Phi - \mathcal{L} \\ &= (\partial_t \Phi)^2 - \frac{1}{2}[(\partial_t \Phi)^2 - (\partial_x \Phi)^2 + \beta \Phi^2 - \lambda \Phi^4] \\ &= \frac{1}{2}[(\partial_t \Phi)^2 + (\partial_x \Phi)^2 - \beta \Phi^2 + \lambda \Phi^4]\end{aligned}$$

Exercise 3: $\beta < 0$ and $\lambda = 0$

- Equation 3.7 (standard massive scalar field theory)

$$\mathcal{L} = \frac{1}{2}[(\partial_t \Phi)^2 + (\partial_x \Phi)^2 - m^2 \Phi^2]$$

- Lagrangian with $\lambda = 0$

$$\mathcal{L} = \frac{1}{2}[(\partial_t \Phi)^2 + (\partial_x \Phi)^2 + \beta \Phi^2]$$

So $\beta = -m^2$, which shows that $\beta < 0$.

- Hamiltonian

$$\mathcal{H} = \frac{1}{2}[(\partial_t \Phi)^2 + (\partial_x \Phi)^2 - \beta \Phi^2] = \frac{1}{2}[(\partial_t \Phi)^2 + (\partial_x \Phi)^2 + m^2 \Phi^2] \geq 0$$

and the minimum is at $\Phi(t, x) = 0$.

Exercise 3: $\beta > 0$ and $\lambda = 0$

- Let $\Phi(t, x) = C$. Then $\mathcal{H} = -\frac{\beta}{2} C^2$. There is no minimum because C can be chosen arbitrarily large.
- That Lagrangian is physically unacceptable.
- Notice that the Hamiltonian is unbounded from below, no matter how small (and positive) β is. The system will exhibit instability.

Exercise 4: Ranges of β and λ for physically acceptable Hamiltonians

- Since the minimum kinetic energy occurs when $\Phi(t, x)$ is a real constant C, it suffices to examine the behavior of the function $\mathcal{H}(C) = \frac{1}{2}[-\beta C^2 + \lambda C^4]$.
- Find extrema: Solve $\partial_C(-\beta C^2 + \lambda C^4) = 0$, i.e., $(-\beta + 2\lambda C^2)C = 0$.
- If $\frac{\lambda}{\beta} \leq 0$ then there is one extremum, $C = 0$. If $\beta < 0$, that extremum is a minimum. See Figure 1.
- If $\frac{\lambda}{\beta} > 0$ then there are 3 extrema. If $\beta > 0$, the Hamiltonian is bounded from below. See Figure 2.
  - This is interesting because the lowest energies are at $\Phi \neq 0$. In such a theory, things tend to settle down into one of the two minima, thus 'breaking' the symmetry around the middle.

Figure 1

Figure: $\beta = -2$, $\lambda = 2$

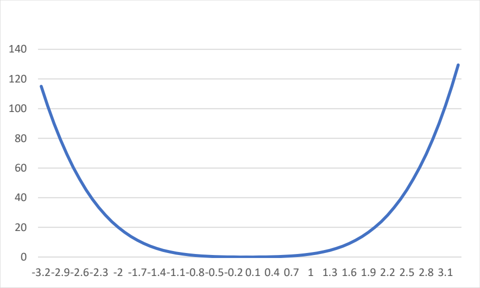

Figure 2

Figure: $\beta = 2$, $\lambda = 2$