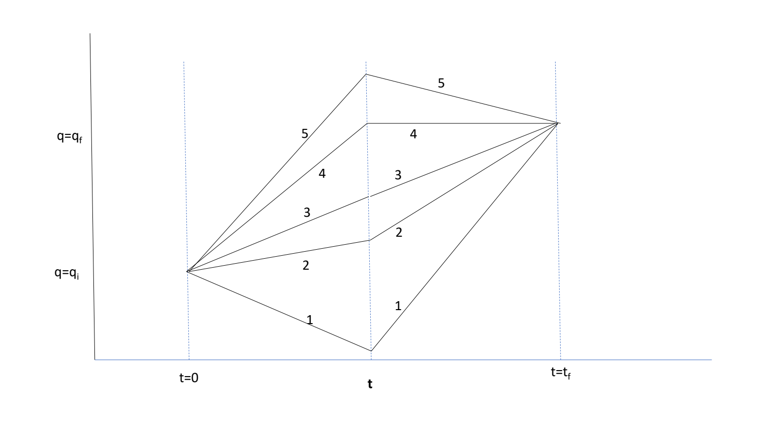
Kachelriess Equation (2.40) (with t_i=0):

$$K(q_f, q_i, t_f) = N \int \mathcal{D}q(t) e^{i \int_{0,q_i}^{t_f,q_f} L(q,\dot{q})dt'} \tag{1}$$

where the integration bound t, q denotes that the time has value t, and that the path has the value q at time t.

Kachelriess doesn't include the constant N. This constant is independent of q_f and q_i and is formally infinite!

Figure: Example paths



Simple example: Set up the integral

The path integral can be approximated by summing over paths as in the Figure.

$$e^{i\int_{0,q_i}^{t_f,q_f} L(q,\dot{q})dt'} = e^{i\int_{0,q_i}^{t,q} L(\hat{q},\dot{\hat{q}})dt' + i\int_{t,q}^{t_f,q_f} L(\hat{q},\dot{\hat{q}})dt'} \tag{2}$$

($\hat{q}$ d a straight line segment to or from time t.) The sum over paths becomes an integral over q at time t.

Simple example: Free theory

Consider

$$L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2$$

Insert into equation (2).

$$e^{i \int_{0,q_i}^{t_f,q_f} \frac{1}{2} m \dot{q}^2} = e^{i \frac{m}{2} \frac{(q-q_i)^2}{t} + i \frac{m}{2} \frac{(q_f-q)^2}{t_f - t}}$$

The exponent is quadratic in q, so we can integrate in closed form.

$$\int e^{i \int_{0,q_i}^{t_f,q_f} \frac{1}{2} m \dot{q}^2} dq = \frac{2\pi i}{m} \sqrt{t(t_f - t)} \sqrt{\frac{m}{2\pi i t_f}} e^{i \frac{m}{2} \frac{(q_f - q_i)^2}{t_f}} \tag{3}$$

$$= \mathcal{N} K(q_f, q_i, t_f)$$

Up to a normalization factor, this agrees with equation (1).

Simple example: Conclusions

For a free theory, the propagator equals (up to a constant) the path integral over paths shown in the Figure. **From Problem 2.1, this is e^{iS} where S is evaluated on the classical path.**

For **general theories**, this is an **approximation** which gets ever better as you break the paths up into more segments (multiple integrals). The approximation is known as Lattice Quantum Mechanics

Methods of stationary phase and steepest descent

Re-introduce $\hbar$ in equation (1) and then analytically continue in time.

$$\hat{K}(q_f, q_i, t_f) = N \int \mathcal{D}q(t) e^{-\frac{1}{\hbar}\int_{0,q_i}^{t_f,q_f} L(q,\dot{q})dt'}$$

If $\hbar$ is small compared to the dimensional quantities of interest, then the integral will be dominated by the contribution(s) from the path(s) which minimize the exponent. **Those paths satisfy the Euler-Lagrange equations, i.e., the classical equations of motion.**

When we don't analytically continue, but leave the factor *i* in the integrand, then the integral is dominated by paths for which the exponent (the phase) is an extremum. Nearby paths have phases which cancel each other.