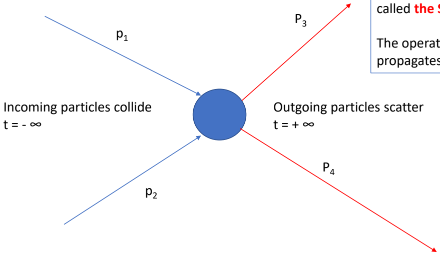
Figure: Scattering



The probability amplitude that two particles, with momenta p_1 and p_2 , collide and end up with momenta p_3 and p_4 is **$< p_3\, p_4\, |S|\, p_1\, p_2 >$** and is called **the Scattering matrix**.

The operator **S** is the evolution operator that propagates the initial state to the final state.

Preview and Motivation for Kachelriess 2.3 – Bill Celmaster, August 2020

Much of Quantum Field Theory is concerned with computing the Scattering Matrix $\langle f|S|i\rangle$.

LSZ Theorem for the Figure

$$\langle p_3 p_4|S|p_1 p_2\rangle = [i\int d^4x_1 e^{-ip_1\cdot x_1}(\Box_1 + m^2)]...[i\int d^4x_4 e^{ip_4\cdot x_4}(\Box_4 + m^2)]$$
$$x\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle$$

where $\Box_n = \frac{\partial^2}{\partial t_n^2} - \Delta_n$. $\phi(x)$ is the field, and the item in red is the Green function. **This is the reason we need to compute the Green functions.**

Kachelriess Section 2.3

A Guide to Reading this Section

The object of Section 2.3 is to arrive at equation 2.55. Steps to unpacking this equation:

- Preliminaries: Heisenberg Picture, Time-Ordered Operator Products
- Comments about the N-point Green function (equation 2.55)
- Transition matrix elements of time-ordered operator products as path integrals
- N-point Green functions versus transition matrices as path integrals
- Green function path integrals and $i\epsilon$
- Computational trick: expressing the N-point Green function as an N^{th} derivative of a simpler path integral

Preliminaries

- We use the Heisenberg Picture. States are time-independent. Operators are time-dependent. E.g.

$$\hat{q}(t) \equiv e^{iHt} \hat{q} e^{-iHt}$$

- Operators at different times generally don't commute. E.g. $\hat{q}(t)\hat{q}(t') \neq \hat{q}(t')\hat{q}(t)$. Define the time-ordered product $T\{\hat{q}(t)\hat{q}(t')\}$ as

$$\begin{aligned} T\{\hat{q}(t)\hat{q}(t')\} &= \hat{q}(t)\hat{q}(t') \qquad \text{if} \quad t > t' \\ &= \hat{q}(t')\hat{q}(t) \qquad \text{if} \quad t' > t \end{aligned}$$

  This can be generalized to more operators, so that they are always arranged with time increasing from right to left.

N-point Green function

- This equation *defines* the N-point Green function. We need to explain the RHS.

$$G(t_1, t_2, ...t_N) \equiv \langle 0| T\{\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)\}|0\rangle \tag{1}$$

  where $|0\rangle$ is the vacuum state.
- It is misleading to write, as Kachelriess does, $|0, \infty\rangle$ etc.
- Although G is called a Green function, it has almost nothing to do with what we called Green functions for differential equations.
- Kachelriess does not motivate our reason for being interested in G! It turns out that if you know G, then you know ***everything*** about the theory.

Transition matrix elements of time-ordered operator products as path integrals

- Read and study Kachelriess's section *Time-ordered products of operators and the path integral* (equations 2.41 through 2.45).
- Key result (version of 2.45) for transition matrix

$$\begin{aligned}&\langle q_f, t_f | T\{\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)\} | q_i, t_i\rangle \\ &= N \int \mathcal{D}q(t)\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)e^{iS[q(t)]}\end{aligned} \tag{2}$$

- $|q_i, t_i\rangle$ means "the eigenstate of the operator $\hat{q}(t_i)$ with an eigenvalue of q_i."
- All paths on RHS begin at time t_i with value q_i and end at time t_f with value q_f. [Contrast this later with unconstrained paths for the Green function path integral.]

Green functions versus transition matrices

- Equation (1) has $\langle 0|T\{\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)\}|0\rangle$ and equation (2) has $\langle q_f, t_f|T\{\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)\}|q_i, t_i\rangle$.
- What is the *real* meaning of $|0\rangle$ in equation (1)?
- How can the RHS of (2) be modified so it can be applied with $|0\rangle$ instead of $|q_i, t_i\rangle$ etc?
- *This is highly non-trivial*. I'll give answers below without proof. Kachelriess tries to motivate the answer in his section on *Vacuum Persistence Amplitudes*. I recommend you skip that!

Green function path integrals and $i\epsilon$

- $|0\rangle$ is the vacuum state, and is the (often unique) lowest energy eigenstate of the Hamiltonian.
- To force causality and also end up with a well-defined Green function

$$\begin{aligned} &\langle 0|T\{\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)\}|0\rangle \\ &= N' \lim_{\epsilon\to 0^+} \int \mathcal{D}q(t)\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)e^{i(S[q(t)]+i\epsilon)} \end{aligned} \tag{3}$$

  where paths asymptotically have constant velocity (as $t \to \pm\infty$) but are otherwise unconstrained.[Contrast this earlier with constrained paths for the transition matrix path integral.]

The importance of N-point Green functions derives from the LSZ theorem of particle scattering (not explained at this point in Kachelriess). The scattering requirements lead to the $i\epsilon$ definition in the Green function definition. References include Schwartz Chapters 6 and 14 and also Collins *https://arxiv.org/pdf/1904.10923.pdf*

Computational trick

- Example. Suppose you want to calculate

$$Q(m,n) = \int_0^\infty \int_0^\infty x^m y^n e^{i[-x^2-y^2]} dxdy$$

- Trick. Define

$$\mathcal{I}(J_1,J_2) = \int_0^\infty \int_0^\infty e^{i[-x^2-y^2+J_1 x+J_2 y]} dxdy$$

The exponent is quadratic so you can easily compute $I(J_1,J_2)$. Then

$$Q(m,n) = (-i\frac{\partial}{\partial J_1})^m(-i\frac{\partial}{\partial J_2})^n \mathcal{I}(J_1,J_2)|_{J_i=0}$$

Computational trick continued

- In the above trick, replace x by $q(t_1)$, y by $q(t_2)$, J_1 by $J(t_1)$ and J_2 by $J(t_2)$. Then

$$Q(1,1) = \int q(t_1)q(t_2)e^{i[-q(t_1)^2 - q(t_2)^2]} dq(t_1)dq(t_2)$$

$$\mathcal{I}(J) = \int_0^\infty \int_0^\infty e^{i[-q(t_1)^2 - q(t_2))^2 + J(t_1)q(t_1) + J(t_2)q(t_2)]} dq(t_1)dq(t_2)$$

and

$$Q(1,1) = (-i\frac{\partial}{\partial J(t_1)})^m(-i\frac{\partial}{\partial J(t_2)})^n \mathcal{I}(J)|_{J(t_i)=0}$$

- Generalize this. Read Kachelriess from equations 2.47 through 2.49. Then skip to text from equations 2.53 through 2.54. Finally put everything together to get 2.55.

Kachelriess 2.3. The Good and The Bad

- Introduction on page 23. If you find it helpful, great. I think it's obscure.
- Notation $\langle q', t' | q, t \rangle$ confuses me. This isn't just a Kachelriess thing. It means $\langle q' | U(t', t) | q \rangle$ where U is the evolution operator. **However, I don't believe that there's any meaning that can be given to the vector $|q, t\rangle$.**
- Time-ordered products page 24. Pretty good.
- External sources pages 24 and 25. I'm not excited by the motivational discussion. I prefer to think of the source J as a trick. Page 25 is better.
- Vacuum persistence amplitudes on pags 25 and 26. I don't like this section. Skip it. For now, ignore Wick rotations and we'll return to these later.
- The summary on page 26 is pretty good but if you don't get the connection with the Feynman propagator, don't worry about it.