Classical Physics in a nutshell

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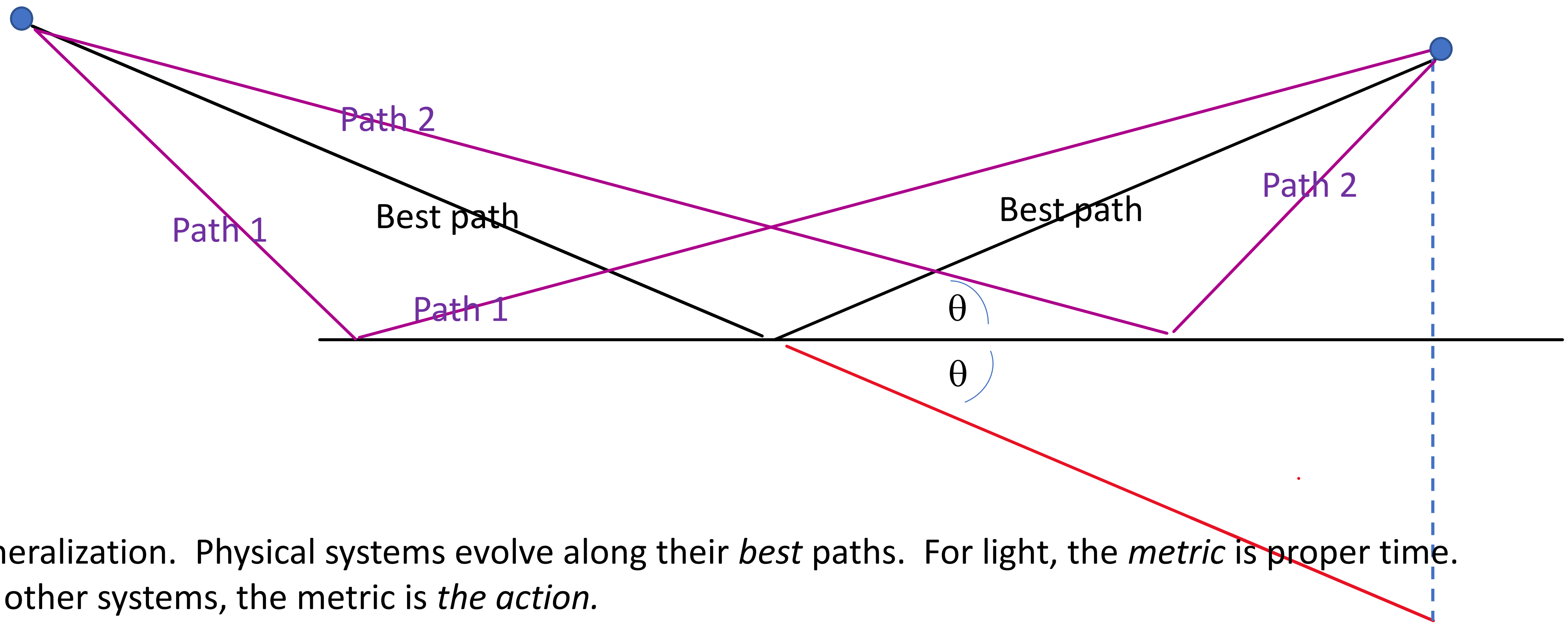
V1.2

Fermat's principle

Light travels along the path that takes the least time. Trick – look at the mirror reflection.
Only the *best* path goes in a straight line through the 'mirror'.

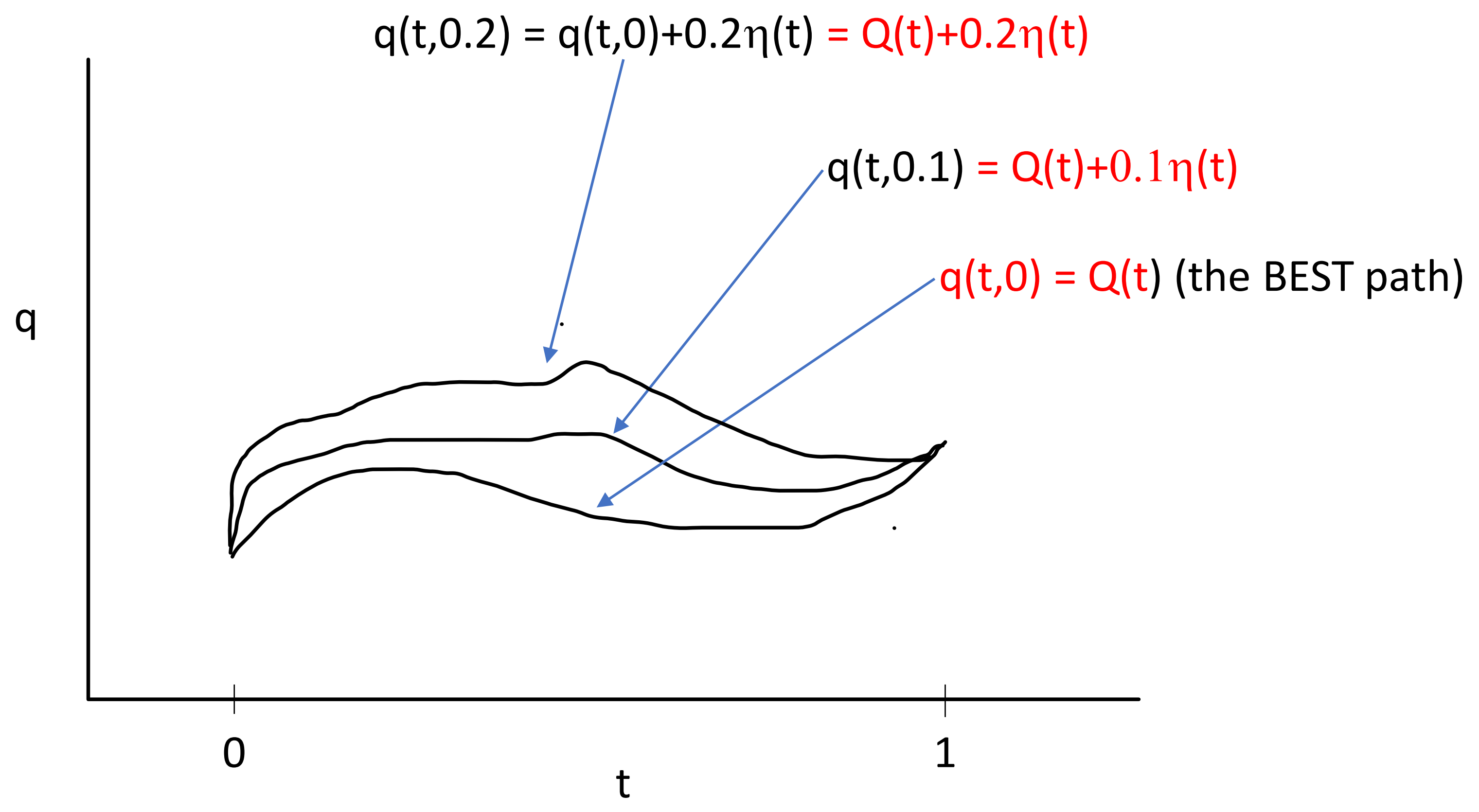


Generalization. Physical systems evolve along their *best* paths. For light, the *metric* is proper time.
For other systems, the metric is *the action.*

Feynman asks "does it *smell* the neighboring paths to find out whether or not they have more action?"
(reference Vol II chapter 19 of Feynman Lectures or https://www.feynmanlectures.caltech.edu/II_19.html)

Classical Mechanics

Example of q(t,ε) with fixed endpoints with very lousy drawing skills.



Compute S(Q+εη). If Q(t) is an extremum, then $\frac{\partial S(Q+\ \varepsilon\eta)}{\partial\varepsilon} = 0$

This is true for each choice of η in which case the integrand of (1.3) is 0.
Leads to the Euler-Lagrange equations (1.4)

Simple example, a spring (linear harmonic oscillator)

The Lagrangian (to be verified shortly) is $\boldsymbol{L = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} k q^2}$

$$S(q) = \int_{T_1}^{T_2} dt\left(\frac{1}{2} m\dot{q}^2 - \frac{1}{2} k q^2\right)$$

Suppose Q is an extremum. Then for all η which are 0 at the boundaries,

$$\frac{dS(Q+\varepsilon\eta)}{d\varepsilon}\Big|_{\varepsilon=0} = 0$$

where

$$S(Q+\varepsilon\eta) = \int dt\left(\frac{1}{2} m\left(\frac{d(Q(t)+\varepsilon\eta(t))}{dt}\right)^2 - \frac{1}{2} k\left(Q(t)+\varepsilon\eta(t)\right)^2\right)$$

$$\frac{dS(Q+\varepsilon\eta)}{d\varepsilon}\Big|_{\varepsilon=0} = \int_{T_1}^{T_2} dt\left(m\left(\frac{d\big(Q(t)\big)}{dt}\right)\left(\frac{d\big(\eta(t)\big)}{dt}\right) - k\big(Q(t)\big)\big(\eta(t)\big)\right)$$

The term with red should be integrated by parts, noting that

$$\left(\frac{d\big(Q(t)\big)}{dt}\right)\left(\frac{d\big(\eta(t)\big)}{dt}\right) = \frac{d\left(\frac{dQ}{dt}\eta\right)}{dt} - \frac{d^2Q}{dt^2}\eta$$

and

$$\int_{T_1}^{T_2} \frac{df}{dt} = f(T_2) - f(T_1)$$

So $$\frac{dS(Q+\varepsilon\eta)}{d\varepsilon}\Big|_{\varepsilon=0} = \int_{T_1}^{T_2} dt\left(-m\frac{d^2Q}{dt^2}\eta - kQ\eta\right)$$

$$= \int_{T_1}^{T_2} dt\left(-m\frac{d^2Q}{dt^2} - kQ\right)\eta$$

Since LHS = 0 for all η we get $$-m\frac{d^2Q}{dt^2} - kQ = 0$$

which is the familiar harmonic oscillator equation. So we have the right Lagrangian.

Important notation stuff (keep this as reference)

- What is the meaning of L(q, $\dot{q}$)
  - Define L(x,y) as a function of 2 parameters x,y then evaluate x at q(t) and y at $\dot{q}$(t).
    So L(q, $\dot{q}$) assigns values to a hypersurface in 2D (x,y)-space. (x and y are both paths)
- What is the meaning of $\frac{\partial L\,(q,\dot{q}\,)}{\partial\,\dot{q}}$?
  - $\frac{\partial L\,(x,y)}{\partial\,y}\,|_{x=q,y=\dot{q}}$
- What is $\delta q(t)$?
  - $\delta q(t)$ = [q(t) + εη(t)]-[q(t)] = **εη(t)**. You keep ε non-zero and η arbitrary until the VERY END of the calculation.
    At the END, you let ε go to 0 and you explore consequences of η being arbitrary.
    Usually, you end up dividing by ε which leads to a proper derivative. Also, you ignore all terms that show up with higher order of ε (e.g. ε^2).
- What is $\delta\dot{q}(t)$?
  - $\delta\dot{q}(t)$ = [$\dot{q}$(t) + ε $\dot{\eta}$ (t)] – [$\dot{q}$(t)] = **ε $\dot{\eta}$ (t)**. Then same interpretation as above.
- What is $\delta O(q)$?
  - $\delta O(q)$ = O(q + εη) – O(q). We write $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\delta O(q)\right) = \int dx\eta(x)\frac{\delta O(q)}{\delta q(x)}$. See Problem 1.6.
- What is δO(q, $\dot{q}$)?
  - δO(q, $\dot{q}$) = O(q + εη,$\dot{q}$(t) + ε $\dot{\eta}$ (t)) - O(q, $\dot{q}$). Then same interpretation as above. Notice
    $O(q+\varepsilon\eta,\dot{q}+\dot{\eta}) - O(q,\dot{q}) = \frac{\partial O(x,y)}{\partial x}(\varepsilon\eta)|_{x=q,y=\dot{q}} + \frac{\partial O(x,y)}{\partial y}(\varepsilon\,\dot{\eta})|_{x=q,y=\dot{q}} = \frac{\partial \boldsymbol{O}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \boldsymbol{q}}\delta\boldsymbol{q} + \frac{\partial \boldsymbol{O}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}}$

Problem 1.6

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\delta O(q)\right) = \int dx\, \eta(x) \frac{\delta O(q)}{\delta q(x)}$$

Take O(q) = q(x). Then

$$\int dx'\, \eta(x') \frac{\delta O(q)}{\delta q(x')} = \int dx'\, \eta(x') \frac{\delta[q(x)]}{\delta q(x')} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[q(x) + \varepsilon\eta(x) - q(x)] = \eta(x)$$

For the two expressions in blue to be equal, $\frac{\delta O(q)}{\delta q(x')}$ must equal $\delta(x-x')$.

Hamilton's equations: Another related approach

Introduce an auxiliary variable, *p*. Then define an action S' which is a function of two variables *q* and *p.* If you do this right, then you find the same extrema as before.

- Step 1 – Define a variable $\check{p} = \frac{\partial L}{\partial \dot{q}}$
- Step 2 – Rewrite the original Lagrangian in terms of $\check{p}$ and *q* as L(*q*, $\check{p}$)
- Step 3 – Define $\breve{H}$ (*q*, $\check{p}$) = $\check{p}\dot{q}$- L(q, $\check{p}$) {usually the t-dependence is only implicit in the variables *q* and $\check{p}$ }.

  Now, define H(q,p) as $\breve{H}$ (*q*, $\check{p}$) but replacing $\dot{q}$ by some function of *q* and $\check{p}$, then substituting *p* for $\check{p}$.
- Step 4 – Define a new action $S_H(q,p) = \int_a^b dt(p\dot{q} - H(q,p))$and find extrema by taking $q = Q + \eta_1, p = P + \eta_2$
- Step 5 – Solve $\dot{q} = \frac{\partial H}{\partial p}$ $\dot{p} = -\frac{\partial H}{\partial q}$

<u>Example: Spring</u>

- Step 1: $\check{p} = \frac{\partial(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2)}{\partial \dot{q}} = m\dot{q}$
- Step 2: $L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 = \frac{\check{p}^{\,2}}{2m} - \frac{kq^2}{2}$, $\tilde{L} = \frac{p^{\,2}}{2m} - \frac{kq^2}{2}$
- Step 3: $H = p\dot{q} - \left(\frac{p^{\,2}}{2m} - \frac{kq^2}{2}\right) = \frac{p^2}{m} - \left(\frac{p^{\,2}}{2m} - \frac{kq^2}{2}\right) = \frac{p^{\,2}}{2m} + \frac{kq^2}{2}$
- Step 4: $S_H(q,p) = \int_a^b dt(p\dot{q} - \frac{p^{\,2}}{2m} + \frac{kq^2}{2})$
- Step 5 (Hamilton's equations):$\frac{dq}{dt} = \frac{\partial H}{\partial p}$ = $\frac{\partial(\frac{p^{\,2}}{2m} + \frac{kq^2}{2})}{\partial p}$ = $\frac{p}{m}$ [same result as step 1];$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\frac{\partial(\frac{p^{\,2}}{2m} + \frac{kq^2}{2})}{\partial q}$ = -kq

  Combine these two equations to get m$\ddot{q}$ = -kq which is the spring equation of motion

Notes

- H is known as the Hamiltonian. In the example, it is positive.
- Generally, H is independent of time.
- Important for the history of quantum mechanics. If we have a function *O(q,p)* then $\frac{dO(q,p)}{dt} = \frac{\partial O}{\partial q}\frac{dq}{dt} + \frac{\partial O}{\partial p}\frac{dp}{dt}$

  Use Hamilton's equations to replace items in red so $\boldsymbol{\frac{dO(q,p)}{dt}}$ = $\frac{\partial O}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial O}{\partial p}\frac{\partial H}{\partial q} \equiv \{\boldsymbol{O}, \boldsymbol{H}\}$

  These are Poisson brackets and become commutators in quantum mechanics.