Kalchereiss Chapters 1.2 and 1.3

Bill Celmaster

June 2020

Where all this is heading

Path-integral approach is based on ordinary complex functions and *not* operators.

Basic tools are

- Equations from classical mechanics, especially involving the Lagrangian and sometimes the Hamiltonian
- Methods of differential equations, especially
  - Linear, second-order with small higher-order terms
  - Perturbative expansions for the higher-order terms
  - Green's functions for a delta-function source
  - Correlation functions as derived from functional dependence on general source terms
- What we won't need (*except for initial setup of the path-integral formalism*)
  - Operator theory
  - Commutation relations
  - Eigenvectors and eigenvalues
- Really???
  - Nahhhh …. But for many of the interesting results of field theory, this 'traditional' quantum mechanics stuff takes a back seat

Green's functions are useful for solving Euler-Lagrange equations

(non-rigorous, non-engineering)

Notation: $\Box\varphi \equiv \frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2}$

Simple equation: $\Box\varphi = J$

Formal solution: $\varphi = \Box^{-1}J$; $\Box^{-1}$ is called a Green's function. Often, J is a delta-function.

Perturbation theory (preliminary to *Feynman diagrams*)

Add a small term to the simple equation. $\Box\varphi(\lambda,t,x) - \lambda\varphi^2(\lambda,t,x) = J(t,x)$. The solution depends on the small parameter λ.

Expand the solution in powers of λ. $\varphi = \varphi(\lambda,t,x) = \varphi_0(t,x) + \lambda\varphi_1(t,x) + \lambda^2\varphi_2(t,x) + \cdots$

The general solution is $\varphi(\lambda,t,x) = \Box^{-1}[J(t,x) + \lambda\varphi^2(\lambda,t,x)]$

$\varphi_0(t,x) = \varphi|_{\lambda=0} = \varphi(0,t,x)$; solution is $\varphi_0 = \Box^{-1}J$

$\varphi_1(t,x) = \frac{\partial\varphi}{\partial\lambda}|_{\lambda=0} = \frac{\partial}{\partial\lambda}\Box^{-1}[J(t,x) + \lambda\varphi^2(\lambda,t,x)]|_{\lambda=0} = \frac{\partial}{\partial\lambda}\Box^{-1}[\lambda\varphi^2(\lambda,t,x)]|_{\lambda=0} = \Box^{-1}[\varphi^2(0,t,x)] = \Box^{-1}[(\varphi_0)^2(t,x)]$

So $\varphi_1(t,x) = \Box^{-1}[\Box^{-1}J\Box^{-1}J]$

$2\varphi_2 = \frac{\partial^2\varphi}{\partial\lambda^2}|_{\lambda=0} = \frac{\partial}{\partial\lambda^2}\Box^{-1}[J(t,x) + \lambda\varphi^2(\lambda,t,x)]|_{\lambda=0} = \Box^{-1}[4\varphi_0(t,x)\varphi_1(t,x)]$

So $\varphi_2 = 2\Box^{-1}(\Box^{-1}J\Box^{-1}[\Box^{-1}J\Box^{-1}J])$

What is $\Box^{-1}$? It's easiest to introduce this for a 1D problem.

(Following Kachelriess with m=1)

Notation: $Dx \equiv \frac{d^2x}{dt^2} + \omega^2 x$

Simple equation: $Dx = J$

Formal solution: $x = D^{-1}J$; D^{-1} is called a Green's function and written as G. Often, J is a delta-function.

What does $x = D^{-1}J$ mean? $x(t) = \int dt'\, D^{-1}(t,t')J(t')$ It's like matrix multiplication.

Kachelriess derives D^{-1} (equation 1.34) *(I use D^{-1} instead of G).* $D^{-1}(t,t') = \int \frac{d\Omega}{2\pi} \frac{e^{-i\Omega(t-t')}}{\omega^2-\Omega^2}$

First verify this is right. **We want to show that $DD^{-1}J = J$.**

$$D\,(D^{-1}J) = \left(\frac{d^2}{dt^2} + \omega^2\right) \int dt' \int \frac{d\Omega}{2\pi} \frac{e^{-i\Omega(t-t')}}{\omega^2-\Omega^2} J(t') = \int dt' \int \frac{d\Omega}{2\pi} \frac{(\omega^2-\Omega^2)e^{-i\Omega(t-t')}}{\omega^2-\Omega^2} J(t') = \int dt' \delta(t'-t)J(t') = J(t)$$

using $\frac{d^2}{dt^2} e^{-i\Omega(t-t')} = -\Omega^2 e^{-i\Omega(t-t')}$ and $\int \frac{d\Omega}{2\pi} e^{-i\Omega(t-t')} = \delta(t-t')$

Unfortunately, $\boldsymbol{D^{-1}(t,t') = \int \frac{d\Omega}{2\pi} \frac{e^{-i\Omega(t-t')}}{\omega^2-\Omega^2}}$ **is NOT well-defined! The integrand diverges at $\boldsymbol{\Omega = \pm\omega}$.**

Resolve this by adding $\pm i\varepsilon$ to the denominator, integrating with Cauchy's residue theorem and taking ε to 0. Still satisfies $\boldsymbol{DD^{-1}J = J}$ but you get different D^{-1} (negative leads to the retarded Green's function) depending on the sign of $\pm i\varepsilon$ and thus different solutions. Impose causality to pick the right sign (retarded).

Problem 1.8a

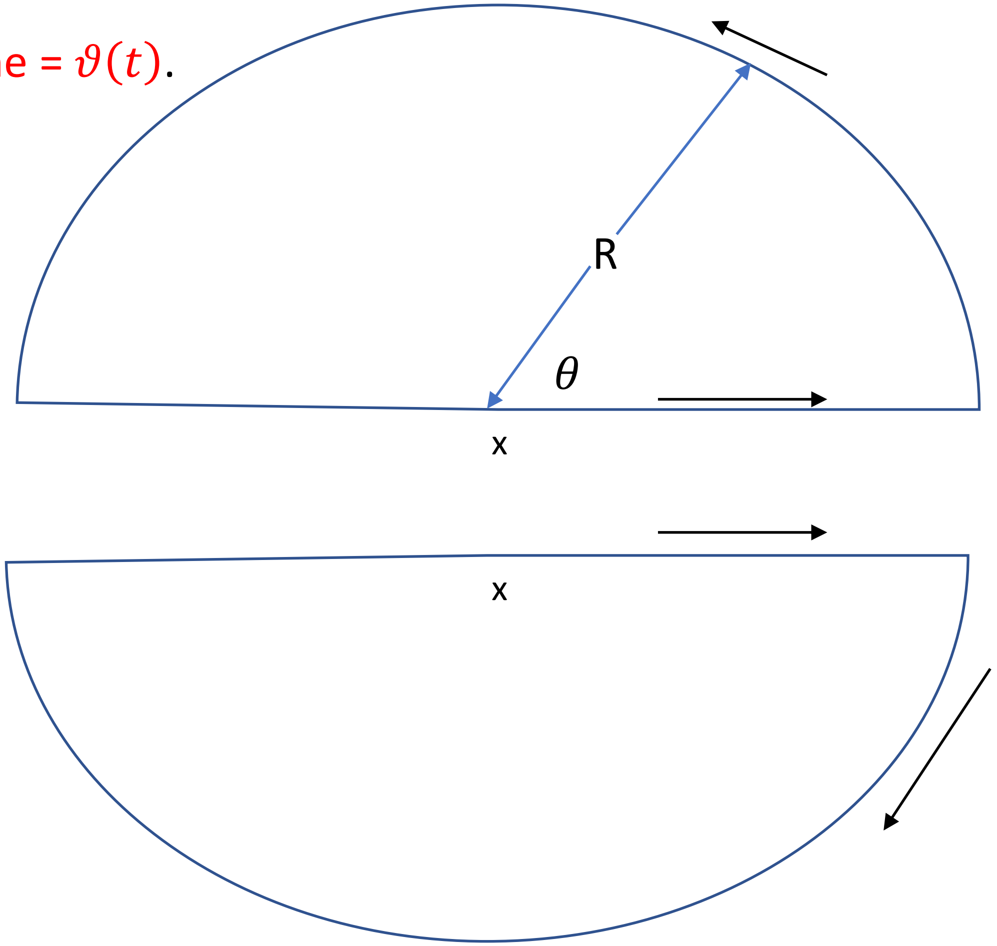
Show $\vartheta(t) = -\frac{1}{2\pi i}\lim_{\varepsilon\to 0+}\int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega t}}{\omega+i\varepsilon}$ where $\vartheta(t)$ =0 if t < 0 and $\vartheta(t)$ =1 if t > 0

Calculus of residues: **t < 0**, choose upper contour.

$$\frac{e^{-i\omega t}}{\omega} = \frac{e^{i|t|R(cos\theta+i\,sin\theta)}}{R(cos\theta+isin\theta)} = \frac{e^{i|t|Rcos\theta}e^{-|t|R\,sin\theta}}{R(cos\theta+isin\theta)}$$ goes to 0 as R → ∞

The integral on the entire semicircle (R → ∞) = integral on real line = $\vartheta(t)$.

But there is no pole within the semicircle so total integral = 0.



When **t > 0**, choose the lower contour and as before, the integral on the curve is 0. However, there is a pole in the semicircle. The residue at $\omega = -i\varepsilon$ is $e^{-\varepsilon t}$ Cauchy's theorem says the integral's value is so

$$\vartheta(t) = \frac{-1}{2\pi i}\lim_{\varepsilon\to 0+}(-2\pi i)\,e^{-\varepsilon t} = 1$$

Takeaways

- Perturbation theory has lots of terms with $\Box^{-1}$. Turns out you can draw diagrams where J is a vertex and $\Box^{-1}$ is a line. (*Like Feynman diagrams*)
- Each $\Box^{-1}$ involves an integral with a quadratic term in the denominator. (*Like Feynman propagator*)
- You need to pick which $\Box^{-1}$ you want. Requires adding $\pm i\varepsilon$ to the denominator. Causality implies $+\, i\varepsilon$.

Relativity

$x^{\mu} = (t, x, y, z)$
$x_{\mu} = (t, -x, -y, -z)$

Frame transformations for motion in the x-direction are

$x'(x, y, z, t) = \gamma(v)(x + vt)$ … where $\gamma(v) = \frac{1}{\sqrt{1-v^2}}$
$t'(x, y, z, t) = \gamma(v)(t + vx)$

y'(x,y,z,t) = y

z'(x,y,z,t) = z

$x^{\mu}x_{\mu} \equiv \sum_{\mu=0}^{\mu=3} x^{\mu}x_{\mu}$ is frame-invariant. Similarly with other *4-vectors.*

This can be generalized to multi-index objects.

Our senses are frame-invariant so **ultimately** the things we measure must be frame-invariant. Such variables are called **scalars.** $x^{\mu}x_{\mu}$ is a scalar.