Lancaster Exercise 10.4

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First, to reiterate what I spoke about at lunch and what I mentioned earlier in emails: usually, the best thing to do in standard (non-gravitational) field theory is to ignore whether indices are upper or lower. On occasion – as in Exercise 10.4 – you might be shown the components of a tensor, and then it's a matter of convention whether the index is upper or lower and whether a minus-sign shows up or not. In virtually any calculation of observable quantities, indices come in pairs and are summed using the convention $a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3$. ***However***, it's inconvenient to have a different convention than the text. A good example is this exercise, where tensors (which the exercise asks about) would differ from Lancaster by overall factors of (-1) if you totally ignored the positions of the indices.

I'll illustrate the derivation of equations (10.44) through (10.46) by looking at specific indices. First, expand the Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. This is a sum which includes the terms $-\frac{1}{4}[-F_{01}F_{01} - F_{10}F_{10}]$. Notice (a) I'm showing all indices here as lower indices and (b) there is a minus sign for each summand coming from the fact that the contraction involves one spacial index. (If we'd been looking at the term $F_{12}F_{12}$ etc. the sign would have been positive because the contraction involves two spacial indices (1 and 2) each contributing a minus sign.)

The *canonical* variables aren't the $F_{\mu\nu}$ but are instead the A_μ. In case you're wondering why, the answer is "*because we say so*" There are actually deeper answers but I don't think any of them are especially obvious. Anyway, let's rewrite the $(0,1)$ terms above using $F_{01} = \partial_0 A_1 - \partial_1 A_0$.

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}[-F_{01}F_{01} - F_{10}F_{10}] + ... \\ &= \frac{1}{4}[2\partial_0 A_1 \partial_0 A_1 + 2\partial_1 A_0 \partial_1 A_0 - 4\partial_0 A_1 \partial_1 A_0] + ...\end{aligned} \tag{1}$$

It should be noticed here and elsewhere the arrangement of upper and lower indices in the expression $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. I'm going to be careful about upper and lower indices in what follows, so that we end up with the same answer as Lancaster.

Take the expansion of (1) to obtain Π^{01}.

$$\begin{aligned}
\Pi^{01} &= \frac{\mathcal{L}}{\partial(\partial_0 A_1)} \\
&= \frac{1}{4}(4\partial_0 A_1 - 4\partial_1 A_0) \\
&= F_{01}
\end{aligned} \tag{2}$$

Notice that on the left there are upper indices, and on the right, derivatives have been taken with respect to lower-indexed objects. This was explained above.

Now $F_{01} = -F^{01}$ because the tensor has one spacial index. So that gives us Lancaster's equation 10.44, as $\Pi^{01} = -F^{01}$, etc.

The next step is to examine the energy-momentum tensor. That tensor was introduced in Lancaster after equation 10.27. In that part of the text, Lancaster was only examining a single field ϕ whereas in this exercise there are 4 fields A_μ. In exercise 10.2, Lancaster asks us to make some generalizations to Lagrangians that have multiple fields. From this you should be able to derive equation 10.45. However, you can also accept 10.45 and go on to show Lancaster equation 10.46.

This is straightforward, simply substituting 10.44 into the first term in 10.45 and 10.43 into the second term in 10.45. The one potential gotcha has to do with the factor δ^μ_ν. As it happens, this has the standard meaning of the Kronecker delta. Namely, if indices are equal then the value is 1 otherwise it's 0. But it's more helpful, in the context of covariant math, to remember that $\delta^\mu_\nu = g^\mu_\nu$. The right-hand side is a Lorentz tensor (meaning it transforms correctly under Lorentz transformations) (whereas the Kronecker delta is simply a matrix with no transformation properties under Lorentz transformations.) Then by the magic of index-positioning and index-contraction (magic that is only valid when dealing with Lorentz vectors and tensors) you can move indices up and down as you please in 10.46, but remembering that contractions always have to involve one upper and one lower.

Now, to get to 10.47, again focus on the $(0,1)$ tensor component and look at the first term on the RHS of 10.46.

$$\begin{aligned}
-F^{0\sigma}\partial^1 A_\sigma + \partial_\lambda X^{\lambda 01} &= -F^{0\sigma}\partial^1 A_\sigma + \partial_\lambda(F^{0\lambda}A^1) \\
&= (\partial_\sigma F^{0\sigma})A^1 + F^{0\sigma}(\partial_\sigma A^1) - F^{0\sigma}\partial^1 A_\sigma \\
&= (\partial_\sigma F^{0\sigma})A^1 + F^{0\sigma}(\partial_\sigma A^1 - \partial^1 A_\sigma) \\
&= (\partial_\sigma F^{0\sigma})A^1 + F^{0\sigma}F_\sigma{}^1
\end{aligned} \tag{3}$$

Here we've noticed that λ is a dummy summation index which can be substituted by σ. In order to end up deriving 10.47, we need to deal with the term $(\partial_\sigma F^{0\sigma})A^1$ Fortunately, the factor in parentheses is 0 precisely because of the Euler-Lagrange equation. The Lagrangian only has factors that are derivatives of the field, so $\frac{\partial \mathcal{L}}{\partial A_\mu} = 0$ resulting in $\partial_\mu \Pi^{\mu\nu} = 0$.