Lancaster Exercise 9.4

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1 Preamble

It seems to me that Lancaster has been imprecise in his notation. He uses the notation **J** and J^i to refer to

- the 4 x 4 matrix representation of rotation-generators (first dimension refers to time, so if you want to, think of it as 3 x 3 in the spacial dimensions)
- the operators which act on functions by effectively changing their arguments by an infinitesimal rotation (said differently, an operator representation of rotation-generators where the operators act on functions and not matrices)

Where this distinction matters is, for example, in equation (9.62). There you see that the book shows J^i is a sum of terms that look like $x^\mu \partial^\nu$. Those are operators acting on functions. They have the effect of changing the arguments of the functions by infinitesimal rotations (and then when we get to (9.64), we get to see how those operators can be exponentiated to produce new operators that change the arguments by finite – i.e. non-infinitesimal – amounts). On the other hand, if you look at equations (9.33) and (9.34), the book shows J^i is a 4 x 4 matrix. Those two descriptions/definitions of J^i are **NOT** the same! Once you get very comfortable with all of this stuff, then you can distinguish between types of generators, based on context. But Lancaster should, in his book, have used separate notation for the two types of rotation generator.

Another comment before actually digging into this problem: Don't get hung up on the appropriate use of upper and lower indices. In Minkowski space (but not in general relativity) the metric tensor is very simple – in fact, it is the inverse of itself which makes life easy. All you need to do is to make sure that both sides of an equation have the same upper and lower indices. If one side has different indices than the other, then those indices better come in pairs (one upper and one lower) and that denotes a summation (otherwise known as a contraction).

2 Stuff from exercise 9.3

For exercise 9.4, we need results from 9.3. I think that the least confusing way of obtaining equation (9.56) is to refer back to Example 9.5 on page 84. There we obtain J^x, J^y and J^z where

$$J^i = -\frac{1}{i}\frac{\partial \mathbf{R}(\theta^i)}{\partial \theta^i} \tag{1}$$

which implies that

$$\mathbf{R}(\theta^i) \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - i\theta^i J^i \tag{2}$$

for small values of θ^i. Then using the expressions (9.33) and (9.34) for J^i we obtain equation (9.56). A similar kind of thing gives you equation (9.55) and then ultimately (9.57)

3 Diving into 9.4

So now we have an expression for how a very small rotation and a very small boost will act on a vector x^μ. Namely, x^μ becomes $x^\mu + \sum_\nu \omega^\mu_\nu x^\nu$. If you also want to act with a small displacement, you add a^μ. So

$$f(x^\mu) = f(x^\mu + a^\mu + \sum_\nu \omega^\mu_\nu x^\nu) \tag{3}$$

and since both a^μ and ω^μ_ν are small, we can Taylor expand to obtain the second equation of (9.59). Then, since $\omega^\mu_\nu = -\omega^\nu_\mu$ (as can be seen from inspecting (9.57), the term $\sum_{\nu,\mu} \omega^\mu_\nu x^\nu \partial_\mu f(x)$ becomes $\frac{1}{2}[\sum_{\nu,\mu}(\omega^\mu_\nu - \omega^\nu_\mu) x^\nu \partial_\mu f(x)]$. The second term in the sum can be rewritten by changing the summation variables, namely $\frac{1}{2}\sum_{\nu,\mu}(-\omega^\nu_\mu) x^\nu \partial_\mu f(x) = \frac{1}{2}\sum_{\mu,\nu}(-\omega^\mu_\nu) x^\mu \partial_\nu f(x)$ and we then end up with equation (9.60).

In equation (9.61), Lancaster simply rewrites (9.60), by defining the matrix $M^{\mu\nu}$ to be the multiplicand of $\omega_{\mu\nu}$ in (9.60) and noticing (see (9.11) and (9.12)) that $p_\mu = i\partial_\mu$.

Next, we consider (9.62) and (9.63). This is where we could potentially get messed up by the definitions of **J** and **K**, as discussed in the preamble. So let's use slightly different notation. Define

$$\begin{aligned} \mathcal{J}^i &= \frac{1}{2}\sum_{jk} \epsilon^{ijk} M^{jk} \\ \mathcal{K}^i &= M^{0i} \end{aligned} \tag{4}$$

where the ϵ symbol denotes as usual, the antisymmetric object whose value is 1 when all indices appear in cyclic order and 0 when two indices are the same. Notice that Lancaster in equation (9.62) has all upper indices but intends repeated indices to be summed-over.

Regard my equations (4) as definitions of $\mathcal{J}$ and $\mathcal{K}$. Now recall from (9.50) that in general, we have a representation of Lorentz transformations that is supposed to look like

$$Rep(\mathbf{\Lambda}) = e^{-i(\mathcal{J}'\cdot\theta - \mathcal{K}'\cdot\phi)} \tag{5}$$

I have purposely modified the equation in Lancaster (introducing notation $Rep(\Lambda)$) to make it clear (again, see the preamble) that what is meant here, is 'the representation of Λ', by which I mean "how the Lorentz transformation Λ operates, i.e. how it causes something to change". In this expression, we have yet to determine the forms of $\mathcal{J}'$ and $\mathcal{K}'$. In the case where our operators are acting on functions so that their space-time arguments transform under Lorentz transformations, we get the transformed function Φ'

$$\mathbf{\Phi}'(x) = \mathbf{\Phi}(\Lambda^{-1}x) \tag{6}$$

The operator which takes Φ to Φ' is what I call $Rep(\Lambda)$.

What was derived in equation (9.61) is an expression for infinitesimal transformations (from now on, ignore the contribution proportional to a^μ for translations). If we examine equation (9.64) and remember that we derived it for infinitesimal Lorentz transformations where the argument x is transformed by Λ rather than Λ^{-1}, then we can see that infinitesimally, my equation (6) has become

$$\mathbf{\Phi}'(x) \approx [1 - \frac{i}{2}\sum_{\mu\nu} \omega_{\mu\nu} M^{\mu\nu}]\mathbf{\Phi}(x) \tag{7}$$

We see that if we expand equation (9.64) as a Taylor series where $\omega_{\mu\nu}$ is small, then it agrees with my equation (7). Furthermore, if we explicitly write out the sum $\sum_{\mu\nu} \omega_{\mu\nu} M^{\mu\nu}$, and substitute the definitions of my equations (4), then we see that we end up with the form of my equation (5) but where we set $\mathcal{J}'$ to $\mathcal{J}$, and $\mathcal{K}'$ to $\mathcal{K}$. And that almost completes this problem.

There are a couple of loose ends. First of all, I don't think it's obvious that if you prove the above equalities for infinitesimal quantities, you necessarily have proven them for finite quantities. That takes a bit more effort. As I recall, it's done by showing that both sides of the equations solve the same differential equations, not only at '0' but for other values as well. Also, please observe that in Lancaster equation (9.57) the boosts are described by v^i whereas in the equation (9.50) as well as the end of exercise 9.4, there is a term $\mathbf{\Phi}$. To see how to relate the velocity to $\mathbf{\Phi}$, look at the text between equations (9.41) and (9.43). We have $tanh(\phi^i) = \frac{v^i}{c}$. For small values of the right hand side, the left hand side is approximated by ϕ^i. So, in my equation (5), replace Φ by $\frac{v}{c}$. Now things look a lot more like what we'd get, using equation (9.57) for $\omega_{\mu\nu}$ except that equation (9.57) has v instead of $\frac{v}{c}$. I think this is another error in Lancaster, although one that I frequently make myself. In many treatments, the speed of light c is set to 1. Lancaster doesn't tend to do that. And to be honest, I don't think he defines the term 'boost' so maybe he means to set the rapidity to v rather than $\frac{v}{c}$. I don't know. Anyway, since angles have no dimensions, one would assume that none of the entries in (9.57) have dimensions, so whatever v is, it must be dimensionless.