Lancaster Chapter 10

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Mike made the comment that the Chapter 10 exercises seem unrelated to the material in the text. I was intrigued, so looked into it. As far as I can tell, he's right about Exercise 10.1. I'll comment on that exercise and also on 10.4 (although maybe 10.4 is more closely connected to material in the text).

1 Exercise 10.1

In order to solve this problem, I had to use the property that in a quantum formulation of field theory

$$[\Phi(x), \pi(x')] = i\hbar\delta(x - x') \tag{1}$$

where $\pi(x')$ is defined in equation (10.11). This is analogous to the usual relationship in quantum mechanics that $[Q, P] = i\hbar$. However, I can't seem to find anywhere in the text (prior to this chapter) where equation (1) is shown. The closest to this seems to be in footnote 6 (left margin) of page 94. In the next chapter, see equation (11.6).

Anyway, once you accept this equation, then it's easy to solve exercise 10.1. First take $\alpha = i$ where $1 \leq i \leq 3$

$$\begin{aligned}
[\phi(x), P^i] &= [\Phi(x), \int d^3x' T^{0i}(x')] \\
&= \int d^3x' [\Phi(x), \pi(x')\partial^i\Phi(x')] \\
&= \int d^3x' [\Phi(x), \pi(x')]\partial^i\Phi(x') \\
&= i\hbar \int d^3x' \delta(x - x')\partial^i\Phi(x') \\
&= i\hbar\partial^i\Phi(x)
\end{aligned} \tag{2}$$

In the third line, I used the fact that $\Phi(x)$ commutes with $\partial^i\Phi(x)$ (again not mentioned by Lancaster prior to this exercise as far as I can tell, but take my word for it) and in the fourth line I used my equation (1). Lancaster has evidently set $\hbar$ to 1. As for the case where $\alpha = 0$, things are a bit more complicated because $\Phi(x)$ does **not** commute with $\partial^0\Phi(x)$ and furthermore, there is an extra term on the right hand side, $-g^{00}\mathcal{L}$. I haven't tried to come up with the best solution for this particular case but if any of you try to do this and run into problems, let me know and I'll give it some extra thought.

2 Exercise 10.4

Re-write the Lagrangian of equation (10.43) by expanding F_{uv}

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{3}$$

Then take the derivative as shown in equation (10.44) (if you have trouble getting the right answer, pretend there is only one spacial dimension and then explicitly write out all the indices – the only nontrivial terms are F_{01} and F_{10}). You should get on the right hand side $\partial^\rho A^\sigma - \partial^\sigma A^\rho = -F^{\sigma\rho}$. To obtain equation (10.46) you'll simply need to expand all the terms and show that these add up to the same thing as what you get when you expand out all the terms in equation (10.45). Remember my preamble to Exercise (9.4) where I remind you not to sweat whether indices are upper or lower. In particular, you can re-write equation (10.45) as

$$T^{\mu\nu} = \Pi^{\mu\sigma}\partial^\nu A_\sigma - \delta^{\mu\nu}\mathcal{L} \tag{4}$$

Equation (10.47) should result from explicitly writing out all the stuff that precedes it (I haven't tried it but it looks straightforward – but maybe messy to derive). Finally, to obtain the expressions in terms of the **E** and **B** fields, you have to recall the relation between those fields and $F^{\mu\nu}$ (or if not, use the relationships between **E** and **B** fields and derivatives of the **A** field). I'm not sure where in Lancaster those were given, but presumably somewhere. If not, you'll need to dig it up out of your background in electromagnetism. Let me know if you need a reference.