Reflections on Kachelriess 2.3

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1 Introduction

The discussions that I've had with various people, including those during our last meeting on August 3, have convinced me to approach Section 2.3 with even more care than I had been striving for. There are quite a few things in this chapter that verge on incomprehensible, in some cases because they aren't motivated, and in other cases because terms aren't defined. To be fair to Kachelriess, he's not the only one whose texts have these kinds of problems.

Several of you have noted that Lancaster appears to be clearer about a number of things that we are encountering in Kachelriess. I'm afraid I am also coming to that conclusion. However, what has drawn me to Kachelriess is that he plunges directly into the Path Integral. Lancaster, like many others, postpones that discussion until much later in his book. I'm hoping that once you all make your way through Kachelriess Chapter 2, you'll be able to settle into the Path Integral way of doing things and that you will come to appreciate this approach. But frankly, since I've never actually learned Quantum Field Theory this way, I can't predict what obstacles are ahead of us.

I had decided to try to press the reset button on all this and see if along the way, I could answer questions that have been posed by several of you, or at least point you to material that seems readable. Before subjecting you to all this, I sent my notes to Jose and Matthew and based on their comments have concluded that these notes are far less clarifying that I'd hoped. So, what follows is a re-write where I've put many of the notes into appendices which you can, at your leisure, look up for reference – but with no promises that they'll help. Here is the outline.

- Takeaways that we'll need to progress to the next section
- Products in the path integral, why we care and how we could compute them on a super-duper-computer
- Time-ordering and path integrals

- Appendix A: (Review)Motivation and preview – the S-matrix
- Appendix B: Some quantum mechanics and notation, especially Schrodinger and Heisenberg formulations
- Appendix C: Vacuum to vacuum transitions, Green functions, causality and $i\epsilon$
- Appendix D: Using the generating functional to compute products in the path integral

By the way, I will *not* refer to the section (pp 25 and 26) on the vacuum persistence attitude since I strongly recommend you don't read it.

2 Progressing to the next sections of Kachelriess

Here are the key points that we'll need for the future.

- (see the end of section 2.3) Definition:

$$G(t_1, ...t_n) \equiv \langle 0|T\{Q(t_1)...Q(t_n)\}|0\rangle \tag{1}$$

  The operators $Q(t)$ are the Heisenberg-picture position operators (I use upper-case for operators rather than the caret notation used by Kachelriess). The term $G(t_1, ...t_n)$ is called the n-point Green('s) function (for non-relativistic quantum mechanics).[1] We care about the Green's function because, as explained in Appendix A, it is required for computing the S-matrix[2] and the S-matrix essentially contains everything we care about in Quantum Field Theory.

- (see 2.53) Definition[3]:

$$Z[J] \equiv \int \mathcal{D}q(t) e^{i(S[q(t)]+J(t)q(t)+i\epsilon)} \tag{2}$$

[1] My notation differs a bit from Kachelriess because as I point out in Appendix B, there is no difference between $|0, \infty\rangle$ and $|0\rangle$.

[2] More precisely, we care about the Green's function for quantum fields Φ which, like the Q operator can be time-ordered.

[3] Implicitly we take $\epsilon \to 0$.

- Path Integral

$$\begin{aligned} G(t_1,...,t_n) &= \langle 0|T\{Q(t_1)...Q(t_n)\}|0\rangle \\ &= K' \int \mathcal{D}q(t) q(t_1)...q(t_n) e^{i(S[q(t)]+i\epsilon)} \\ &= K'(-i)^n [\frac{\delta}{\delta J(t_1)} ... \frac{\delta}{\delta J(t_n)}]|_{J(t)=0} \int \mathcal{D}q(t) e^{i(S[q(t)]+J(t)q(t)+i\epsilon)} \\ &= (-i)^n \frac{1}{Z[0]} \frac{\delta^n Z[J]}{\delta J(t_1)...\delta J(t_n)}|_{J(t)=0} \end{aligned} \tag{3}$$

Unlike Kachelriess, I've kept the constant K' which, from equation (15) in Appendix C, is just the inverse of $Z[0]$.

3 Computing the path integral including products

Consider equation (3), with only two insertions ($n = 2$).

$$\langle 0|T\{Q(t_a)Q(t_b)\}|0\rangle = K' \int \mathcal{D}q(t)[q(t_a)q(t_b)e^{i(S[q(t)]+i\epsilon)}] \tag{4}$$

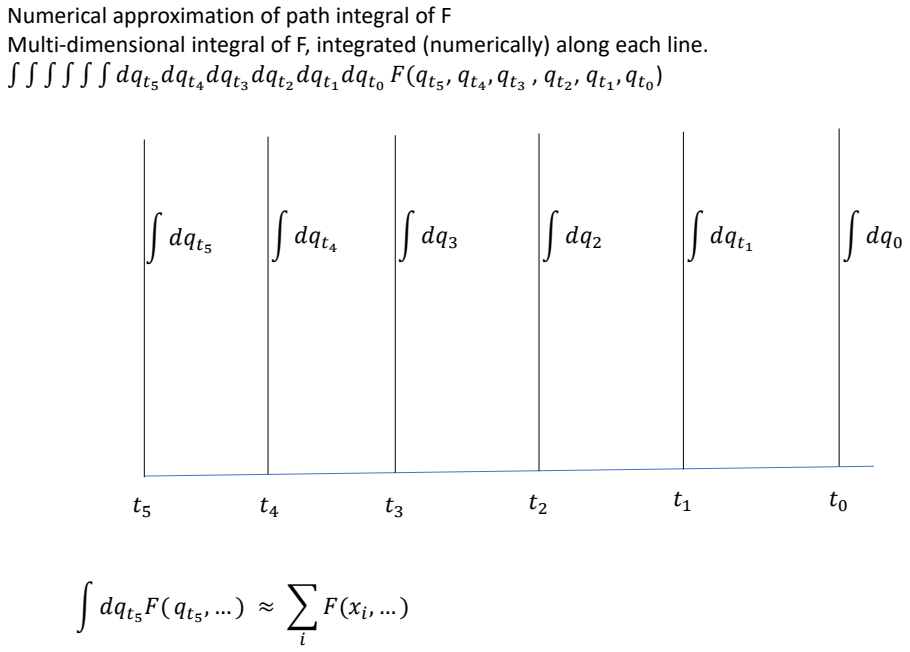
For now, focus on the right hand side without asking whether the equation is correct, or what is the significance of the left hand side. Can we compute it?

The right-hand side is a multi-dimensional integral. To be precise, it is infinite-dimensional, but we can approximate it numerically by picking only a finite number of the integrals and then taking a limit. The numerical process is illustrated in Figure (1) which is an approximation to Equation (4) where we take $F(...) = [q_{t_a} q_{t_b} e^{i(S[q(t)]+i\epsilon)}]$, evaluating the action at the points q_{t_i} where the t_i include t_a and t_b. It is important to notice, is that the integrals can be evaluated in any order. So even though we imagine paths that progress from right to left corresponding to an ordering of time, the actual calculation doesn't depend on this path interpretation. The time-ordering is not encoded in the order that we choose the integrals. Rather, it is encoded in the action S in terms like $\frac{1}{\delta}(q_{t+\delta} - q_t)$ which are finite-difference approximations of $\dot{q}$.

In principle, if everything converged[4] and we had a big enough computer, all of quantum mechanics and quantum field theory could be worked out numerically

[4]What could cause numerical difficulty, is the fact that $e^{i(S+i\epsilon)}$ is complex. However, analyticity properties can be proven and it turns out that by rotating time in the complex plan by 90 degrees ($t \to it$), all Green's functions may be analytically continued as functions of the times. This rotation is called the *Wick rotation*. Kachelriess elaborates on this in Remark 2.1. The Wick rotation results in the path-integral integrands being real (and bounded) rather than complex, and thus amenable to numerical computation.

Figure 1: Numerical approximation to the path integral



by integrating the path integral. That, in fact, has been the approach of Lattice Quantum Field Theory and it has been remarkably successful at calculating, based on the Lagrangian theory for quarks and gluons, masses of elementary particles like the proton and pion, as well as many other measured parameters of the theory.

Numerical methods aren't satisfying ways of gaining insight into the theory and for many problems, they aren't sufficiently accurate. So it's important to develop some intuitive understanding of the path integral. If you have the patience for it, I recommend portions of the book by Feynman and Hibbs. In particular, section 6.1 from about pp. 120 - 125, provides a rather picturesque interpretation of the path integral as applied to a perturbative expansion (actually, it's the Dyson series although they don't say so).

What I **don't** find useful, is Kachelriess's statement on page 25 just prior to his section on the Vacuum Persistence Amplitude. He says, "Physically, the expression (2.50) corresponds to the probability amplitude that a particle moves from $q_i(t_i)$ to $q_f(t_f)$ having the intermediate positions $q(t1), ..., q(t_n)$". The way I understand this, it would seem that the only paths to be considered are those which pass through points mentioned ($q(t1)$, etc.). On the other hand, that doesn't tell us anything since $q(t1), ..., q(t_n)$ can take on any values. So this simply appears

to be the path integral but without the $q(t_i)$ insertions. It seems to me that the significant new feature distinguishing this path integral from the one representing the propagator (as covered in section 2.2), is that each path gets weighted by the values of $q(t_i)$ which it passes through.

4 Time Ordering

Let's make sure we're clear about the definition of time-ordering in equation (1). Operators at different times generally don't commute. E.g. $Q(t)Q(t') \neq Q(t')Q(t)$. Define the time-ordered product $T\{Q(t)Q(t')\}$ as

$$\begin{aligned} T\{Q(t)Q(t')\} &= Q(t)Q(t') \quad \text{if} \quad t > t' \\ &= Q(t')Q(t) \quad \text{if} \quad t' > t \end{aligned}$$

This can be generalized to more operators, so that they are always arranged with time increasing from right to left.

Kachelriess, in the section from equations 2.41 to 2.45, proves equation (4) above. I would suggest that it is easier to understand his proof if you look at the special case with Q instead of A and B. Consider for example, the term $Q(t_a)$. Recall from the derivation of the path integral, that we have a many-dimensional integral, and for each integral, we have a resolution of the identity of the form $\sum |q_n, t_n\rangle\langle q_n, t_n|$. In particular, we have an integral of the form

$$\int dq_a e^{iS[q]} |q_a, t_a\rangle\langle q_a, t_a| \tag{5}$$

Now suppose you multiply the integrand by q_a, obtaining

$$\int dq_a e^{iS[q]} q_a |q_a, t_a\rangle\langle q_a, t_a| \tag{6}$$

Since $Q(t_a)|q_a, t_a\rangle = q_a|q_a, t_a\rangle$, we can instead write the above integral as

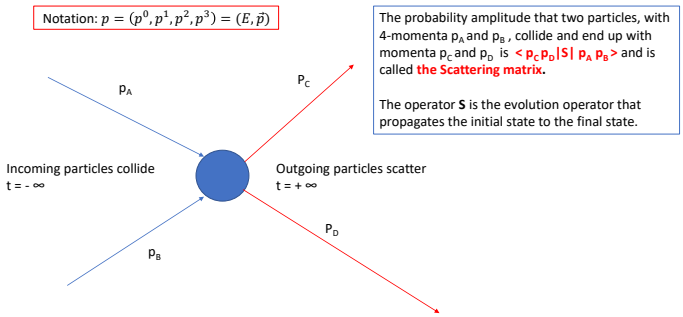
$$Q(t_a) \int dq_a e^{iS[q]} |q_a, t_a\rangle\langle q_a, t_a| \tag{7}$$

Recall that we were only looking at one of the many integrals that make up the path integral, and that each has its own resolution of the identity. This demonstrates that the effect of including q_a as a factor in the integrand, is the same as including a factor of $Q(t_a)$ in the inner product $\langle q_f, t_f|T\{Q(t_a)...\}|q_i, t_i\rangle$. Kachelriess explains, in equation 2.43, why the time-order naturally falls out as a consequence of the ordering of the integrals[5].

[5]More precisely, as I pointed out earlier,what ultimately matters here isn't the order of the integrals, but rather the time-encoding implicit in the action. But by doing the integrals in time-order, the snippets of action become organized in natural time order so this is the easiest way to think about things.

Appendix A: Motivation and Preview

Figure 2: Scattering



Much of Quantum Field Theory is concerned with computing the Scattering Matrix $\langle f|S|i\rangle$. There is a theorem in field theory (section 2.3 is about nonrelativistic quantum mechanics – field theory comes later) called the LSZ theorem (not proven here)]. For Figure (2), the LSZ theorem is

$$\langle p_C p_D|S|p_A p_B\rangle =$$
$$i\int d^4x_A e^{-ip_A\cdot x_A}(\Box_A + m^2)d^4x_B e^{-ip_B\cdot x_B}(\Box_B + m^2)d^4x_C e^{ip_C\cdot x_C}(\Box_C + m^2)]d^4x_D e^{ip_D\cdot x_D}(\Box_D + m^2)$$
$$\langle 0|T\{\phi(x_A)\phi(x_B)\phi(x_C)\phi(x_D)\}|0\rangle \tag{8}$$

where $\Box_A = \frac{\partial^2}{\partial(x^0_A)^2} - \frac{\partial^2}{\partial(x^1_A)^2} - \frac{\partial^2}{\partial(x^2_A)^2} - \frac{\partial^2}{\partial(x^3_A)^2}$, $\phi(x)$ is the field, and the last line is the Green function.

So, the physics we care about is on the left hand side of the equation, and the right hand side is what we need to compute. The goal of Quantum Field Theory is to compute Green functions.

Appendix B: Some quantum mechanics and notation

In bra-ket notation, the quantities we can measure in QM look like

$$\langle s_1|O|s_1\rangle \tag{9}$$

where the ket $|s_1\rangle$ is a vector in the Hilbert space, the bra $\langle s_1|$ is its dual and O is a Hermitian operator that acts on the Hilbert space. The measured quantity is

described as the expectation value of that observable when the system is in the state $|s_1\rangle$.

We also encounter quantities of the type $\langle s_2|O|s_1\rangle$ in intermediate steps of quantum mechanical calculations. Those quantities generally don't have a useful physical interpretation although physicists occasionally come up with pictorial ways of describing those quantities.

If O_1 and O_2 are Hermitian operators that don't commute, then O_1O_2 is an operator which is **not** Hermitian. Therefore it is not observable. However, such an operator might occur at intermediate steps in quantum mechanical calculations.

Often, authors try to distinguish between numbers and operators by using a caret to indicate an operator. For example $\hat{Q}$ would be the position operator. In these notes I mostly won't use the caret to designate operators, but will often use upper case letters for operators and lower case letters for complex numbers.

The laws of physics tell us how expectation values change with time. The laws can be described in various equivalent ways, since only the combination in equation (9) is relevant. There are three well-known *pictures* for describing time-evolution.

1. **The Schrodinger Picture – the Picture generally used for nonrelativistic quantum mechanics**. Parameterize the state with time, t. For example $|s_1(t)\rangle$. Explicitly, a reference time t_0 is chosen, and we write $|s_1(t)\rangle = U(t,t_0)|s_1(t_0)\rangle$ where U is a unitary operator called the evolution operator. The observable represented by O is generally time-independent. Then the expectation value changes from $\langle s_1(t_0)|O|s_1(t_0)\rangle$ to $\langle s_1(t)|O|s_1(t)\rangle$. If we want to avoid ambiguity, we sometimes write $O_{Schrodinger} \equiv O$ and $|s_1(t)_{Schrodinger}\rangle \equiv |s_1(t)\rangle$

    - I mentioned a reference time t_0. Most often, we choose that to be 0. That will be my convention from now on. So $\mathbf{t_0 = 0}$.
    - In the Schrodinger Picture, when an observable can be mathematically represented as $O(t,...)$ we say that the observable is *explicitly time dependent*. This can happen, for example, if our system of interest (described by the Hilbert space) is under the influence of some external time-changing forces. An important case is the Hamiltonian. In most of the Quantum Field Theory we do, the Hamiltonian operator (in the Schrodinger Picture) is <u>not</u> explicitly time-dependent. In that case, $U(t,0) = e^{-iHt}$.

2. **The Heisenberg Picture – the Picture generally used in Quantum Field Theory**. The states are time-independent. For connecting with the Schrodinger picture, set $|s_1\rangle_{Heisenberg} \equiv |s_1(t_0)\rangle_{Schrodinger}$. Now the observables are represented by time-dependent operators $O(t)$. In order for the Schrodinger and Heisenberg pictures to give the same expectation values, we must have $O(t) = e^{iHt}Oe^{-iHt}$. If we want to avoid ambiguity, we sometimes write $O_{Heisenberg}(t) \equiv O(t)$.

3. **The Interaction Picture – the Picture generally used for old-fashioned perturbation theory – included for completeness, and also because I'll briefly mention it later**. This is a hybrid between the Heisenberg and Schrodinger picture and is typically used in situations where it is convenient to write the Schrodinger picture Hamiltonian as a sum $H = H_0 + H_I$ where H_0 is the usual free Hamiltonian (or more generally, the Hamiltonian of an explicitly solvable theory) and H_I is called the interaction term. Then both states and observables are parameterized by time, but with the equations $|s_1(t)\rangle_{interaction} \equiv e^{iH_0t}e^{iHt}|s_1(0)\rangle_{Schrodinger}$ and $O(t)_{interaction} = e^{iH_0t}O_{Schrodinger}e^{-iH_0t}$.

So far, I've only referred to states in the generic form $|s_1\rangle$ or $|s_1(t)\rangle$. But commonly, states are written differently. Here are some examples that we will encounter frequently.

- $|q\rangle$ is a state in the Heisenberg picture, which is an eigenstate of the Heisenberg-picture position operator $Q(0)$, with eigenvalue q. In other words, $Q(0)|q\rangle = q|q\rangle$. (Notice that $Q(0) = Q$, where Q is the Schrodinger-picture position operator.) For sticklers among you, the $|q\rangle$ states aren't strictly speaking states of the Hilbert space since Q doesn't have a discrete spectrum. There are many ways to deal rigorously with this kind of thing. Mathematicians speak of the spectral decomposition. But physicists can get away pretending that these states form some kind of a limit of discrete bonafide Hilbert space vectors. (If there is more than one eigenvector with the same eigenvalue q, some further notation is required to distinguish the vectors from one another. For example $|q, \sigma\rangle$).

- $|p\rangle$ is the state which is an eigenstate of the momentum operator P, whose eigenvalue is p. (Again, recall that the Heisenberg-picture operator $P(0)$ is the same as the Schrodinger-picture operator P.)

- $|\vec{p}\rangle$ is the state which is a simultaneous eigenvalue of operators P_x, P_y, P_z with eigenvalues respectively p_x, p_y, p_z. By the way, this is an example of the kind mentioned above where it would be insufficient to simply specify

$|p_x\rangle$ since there are many P_x vectors whose eigenvalue is p_x but which have different eigenvalues for P_y etc.

- $|\psi_n\rangle$ is often used to designate a state which is the n^{th} eigenstate of some generic operator – often the Hamiltonian.

- $|0\rangle$ and $|\Omega\rangle$ are both used to denote the lowest-energy eigenstate (i.e. the eigenstate of H with the lowest eigenvalue) of the Hamiltonian. Depending on context, we call that the **vacuum** state or the **ground** state.

- The state $|q_a, t_a\rangle$ often arises, and is a somewhat peculiar beast (in fact, I'd persuaded myself that the notation was bogus but I've changed my mind). We say, in English that this state "is the state which, at time t_a has the position q_a". Mathematically, it is a Heisenberg Picture state which is an eigenstate of $Q(t_a)$ with the property that $Q(t_a)|q_a, t_a\rangle = q_a|q_a, t_a\rangle$. Now, from this definition we have

$$e^{iHt_a}Qe^{-iHt_a}|q_a, t_a\rangle = q_a|q_a, t_a\rangle$$

Multiply left and right sides by e^{-iHt} to obtain

$$Q[e^{-iHt_a}|q_a, t_a\rangle] = q_a[e^{-iHt}|q_a, t_a\rangle]$$

where we've invoked the fact that operators like e^{-iHt} commute with numbers like q_a. On both the left and right side, there is a state in square brackets, which we will denote as $|s\rangle$, i.e. $|s\rangle \equiv [e^{-iHt_a}|q_a, t_a\rangle]$. The equation above simply becomes

$$Q|s\rangle = q_a|s\rangle \tag{10}$$

In other words, it is an eigenstate of the position operator Q (or alternatively the Heisenberg position operator $Q(0)$), with eigenvalue q_a. Recall that this is precisely the definition of the Heisenberg-picture state $|q_a\rangle$ (as always, assume there's only a unique such eigenstate). So what we've proven is that

$$e^{-iHt_a}|q_a, t_a\rangle = |q_a\rangle. \tag{11}$$

Recalling that Schrodinger-picture states are related to Heisenberg-picture states by $|s(t)\rangle = e^{-iHt}|s\rangle$, we see that $|q_a\rangle$ is the Schrodinger-picture state evolved for a time t_a from the Heisenberg-picture state $|q_a, t_a\rangle$. In other words, the Schrodinger-picture state which, at $t = 0$ is an eigenstate of Q with eigenvalue q_a has evolved over a time t_a from the state $|q_a, t_a\rangle$. Similarly, we can draw the conclusion that the Schrodinger-picture state $|q_b\rangle$ which at time $t = 0$ is an eigenstate of Q with eigenvalue q_b, evolves over

a time t_b into $|q_b, t_b\rangle$. Alternatively, note that by multiplying both sides of equation (11) with e^{iHt}, we have $|q_a, t_a\rangle = e^{iHt_a}|q_a\rangle$. A similar argument can be applied to $\langle q_b, t_b|$ resulting in $\langle q_b, t_b| = \langle q_b|e^{-iHt_b}$. Putting all this together, we end up with

$$\langle q_b, t_b|q_a, t_a\rangle = \langle q_b|e^{-iH(t_b-t_a)}|q_a\rangle \tag{12}$$

- Following above, we would say that the meaning of $|0, t_a\rangle$ is "the Heisenberg-picture state with the property that it is the lowest-energy eigenstate of the Heisenberg-picture Hamiltonian $H(t)$". You'll see this notation frequently. But why? We have $H(t) \equiv e^{iHt}He^{-iHt}$. But all these operators commute so the exponentials cancel and we have, as expected $H(t) = H$. The ground state of $H(t_a)$ is therefore the ground state of $H(t_b)$ and so $|0, t_a\rangle = |0\rangle$.

  - **YOU DO NOT NEED TO FOLLOW ANY OF WHAT I WRITE IN THIS BULLETED ITEM. I ADD IT ONLY FOR INTEREST BUT NOTHING DEPENDS ON IT.** I believe that this notation persists in the literature owing to its origins in the original scattering theory work by Dyson. He derived the Scattering Matrix by working in the Interaction Picture. An excellent exposition of this is given in Lancaster chapter 18 (Chapter 18.2 pp 167-168 explains the Interaction Picture, Chapters 18.3 and 18.4 explain the Dyson expansion and the importance of time-ordering). The Dyson expansion is an alternative slightly-outdated alternative to the LSZ formulation, but it's arguably a bit easier to follow the proof of the Dyson expansion than the proof of LSZ. In the Interaction Picture, we have split the Hamiltonian into a free part H_0 and an interaction part. You'll see that $H(t)_{interaction}$ is actually time-dependent. So if you are working in the interaction picture, then $|0, t_a\rangle_{interaction}$ isn't $|0\rangle_{interaction}$ and it's important to distinguish the two states. Since we work exclusively in the Heisenberg picture in Kachelriess, I see no value in the notation $|0, t_a\rangle$, although admittedly I find that notation often makes its way into most other texts.

Appendix C: Vacuum to vacuum transitions and causality

Let's review. Kachelriess has introduced, without motivation, a quantity $\langle q_f, t_f|T\{Q(t_a)Q(t_b)\}|q_i, t_i\rangle$ (in his equation 2.45) and then shown that it is equivalent to a certain path integral which was discussed above. With equally no motivation, Kachelriess tells

us in the section entitled "Vacuum persistence amplitude", that we **actually** want the initial and final states to be the Hamiltonian ground state. In other words $\langle 0|T\{Q(t_a)Q(t_b)\}|0\rangle$. Most other treatments of the subject don't bother with the first quantity (from equation 2.45). I have, with the LSZ equation (8), claimed why it is important to examine vacuum-to-vacuum time-ordered products. To remind you, the importance is that it's a mathematical way to compute the scattering matrix. I made no attempt to prove this statement. How does the path integral change when we have an initial state $|0\rangle$ instead of $|q_a, t_a\rangle$? It turns out that a proper answer to this question is a bit involved. Kachelriess makes a weak attempt to discuss this in his section on "Vacuum persistence amplitude", but he simplifies way too much and ultimately does little more than motivate the final answer. I **do not** recommend that you attempt to follow him. If you really want a readable treatment, I recommend Schwartz section 14.4. If you don't have the text, ask me and I'll scan the 2 or 3 pages. In the meantime, I'll just tell you the answer.

$$\langle 0|T\{Q(t_a)Q(t_b)\}|0\rangle = K' \int \mathcal{D}q(t)[q(t_a)q(t_b)e^{i(S[q(t)]+i\epsilon)}] \tag{13}$$

where the paths on the right are unconstrained (they don't begin at a particular point $q_a(t_a)$ or end at a particular point $q_b(t_b)$). K' is a constant (generally different than the constant K in equation (4)) independent of time. Also, by convention, the notation means that we take the limit $\epsilon \to 0^+$. In fact, the path-integral notation we've been using looks the same (other than for the $i\epsilon$) as it did in equation (4) for $\langle q_f, t_f|T\{Q(t_a)Q(t_b)\}|q_i, t_i\rangle$. However, in both cases, the path integral notation was a bit deficient because it doesn't show the end-state constraints.

All of these equations are easily generalized to the case where there are more terms in the time-ordered product, with K' independent of the number of terms that are time-ordered. One special case is when there are no terms in the time-ordered product. Specifically,

$$\langle 0|0\rangle = K' \int \mathcal{D}q(t)e^{i(S[q(t)]+i\epsilon)} \tag{14}$$

But we normalize $\langle 0|0\rangle = 1$. Therefore

$$K' = [\int \mathcal{D}q(t)e^{i(S[q(t)]+i\epsilon)}]^{-1} \tag{15}$$

To summarize, there are two differences in the path integral when we are looking at the vacuum-to-vacuum time-ordered product. First, the paths are unconstrained. Second, there is an $i\epsilon$ introduced into the action. As you may recall from several sections ago, the $i\epsilon$ imposes causality and, as it turns out, is also necessary to make the path integral converge.

One minor point that comes up is this: As you compute the path integral with the $i\epsilon$ insertion, you end up with expressions for the propagator that may involve, for example, $\epsilon\omega_q$ or ϵq^2 etc. There's nothing wrong with that provided that the overall term is positive when ϵ is positive. If so, then those expressions can be rewritten replacing ϵ by ϵq^2 (or whatever) by ϵ' and then replacing $lim_\epsilon \to 0^+$ by $lim_{\epsilon'} \to 0^+$. You might ask why we should go to all that trouble and, in fact, it's not necessary. But by convention, various computed expressions such as the propagator are written with a modification of the form $i\epsilon$ rather than something like $i\epsilon q^2$.

Appendix D: Generating functional

We now talk about what happens if we add to the Lagrangian, a term $J(t)q(t)$ Why do we care? There are two reasons for caring. First, and most important for our purposes, has to do with a well-known mathematical trick which is summarized by the phrase "calculating moments from a generating function". A second reason we might want to add to the Lagrangian a term $J(t)q(t)$, is that we might be interested in a new theory which requires such a term. That would typically be a theory that describes a system which interacts with an external *source* $J(t)$.[6] Such theories were of particular interest to Julian Schwinger (who wrote a text called "Source Theory") and some other physicists, but is of little consequence – other than for some interesting mathematical results – to most current work in quantum field theory.

We focus on the use of the source term to form a generating functional. Here is the general outline of the trick. Suppose you want to calculate

$$Q(m,n) = \int_0^\infty \int_0^\infty x^m y^n e^{i[-x^2-y^2]} dxdy$$

To do that, define

$$\mathcal{I}(J_1,J_2) = \int_0^\infty \int_0^\infty e^{i[-x^2-y^2+J_1x+J_2y]} dxdy$$

The exponent is quadratic so you can easily compute $I(J_1,J_2)$. Then

$$Q(m,n) = (-i\frac{\partial}{\partial J_1})^m (-i\frac{\partial}{\partial J_2})^n \mathcal{I}(J_1,J_2)|_{J_i=0}$$

[6]Since $J(t)$ explicitly depends on the time t, such a Lagrangian leads to a Schrodinger-picture Hamiltonian with an explicit time-dependence. Earlier, I mentioned that in those circumstances, there are equations which need to be modified so that e^{-iHt} is replaced by the more general evolution operator $U(t)$. As it turns out, this won't change the outcomes of any theorems or calculations we care about.

The path integral is considerably more complicated-looking because instead of a 2-dimensional integral over x and y, it is an infinite-dimensional integral over $q(t)$ for all t. Nevertheless it's easy to generalize the above example. In the above trick, replace x by $q(t_1)$, y by $q(t_2)$, J_1 by $J(t_1)$ and J_2 by $J(t_2)$. Then for example

$$Q(1,1) = \int q(t_1)q(t_2)e^{i[-q(t_1)^2-q(t_2)^2]}dq(t_1)dq(t_2)$$

$$\mathcal{I}(J) = \int_0^\infty \int_0^\infty e^{i[-q(t_1)^2-q(t_2))^2+J(t_1)q(t_1)+J(t_2)q(t_2)]}dq(t_1)dq(t_2)$$

and

$$Q(1,1) = (-i\frac{\delta}{\delta J(t_1)})(-i\frac{\delta}{\delta J(t_2)})\mathcal{I}(J)|_{J(t)=0}$$

Note that the derivatives are evaluated for $J(t) = 0$ for <u>all</u> t. The notation $\frac{\delta}{\delta J(t)}$ may be unfamiliar. If so, I recommend that you think of this as $\frac{\partial}{\partial J_i}$ and that you think of $\int J(t)q(t)dt$ as $\sum J_i q_i$. In other words, think of t as an index. Of course, t is a continuous variable, and an index is generally regarded as discrete. So unsurprisingly, you end up replacing quantities like $\delta_{i,j}$ with quantities like $\delta(t-t')$. All of this can be made rigorous using the theory of distributions. But I think it will suffice for you to read Kachelriess starting at the paragraph prior to 2.48 and going through 2.50 where he illustrates how to apply the trick for time-ordered products.

For the vacuum-to-vacuum time-ordered product in equation (13), the generating-functional trick leads to

$$\langle 0|T\{Q(t_a)Q(t_b)\}|0\rangle = K'(-i\frac{\delta}{\delta J(t_a)})(-i\frac{\delta}{\delta J(t_b)})|_{J(t)=0}\int \mathcal{D}q(t)e^{i(S[q(t)]+J(t)q(t)+i\epsilon)} \tag{16}$$

where as usual, the notation implies that we are taking $lim_\epsilon \to 0^+$.