Problem setup – Bill Celmaster, October 2020

- $$\mathcal{L}(\phi, J) = \frac{1}{2}[\partial_\mu \phi \partial^\mu \phi - m\phi^2] - V(\phi) + J\phi$$

  $$Z[J] = \frac{\int \mathcal{D}\phi e^{i \int d^4 x \mathcal{L}(\phi, J)(x)}}{\int \mathcal{D}\phi e^{i \int d^4 x \mathcal{L}(\phi, 0)(x)}}$$

- Z[J] is *NOT* a simple function of J.
- BUT define W[J] = -i ln(Z[J]).
- Recall

  $$\mathcal{G}(x_1, ..., x_n) \equiv \frac{1}{i^n} \frac{\delta^n}{\delta J(x_1)...\delta J(x_n)} Z[J]|_{J=0}$$

  and

  $$G(x_1, ..., x_n) \equiv \frac{1}{i^n} \frac{\delta^n}{\delta J(x_1)...\delta J(x_n)} iW[J]|_{J=0}$$

Problem statement

- Is it necessarily true that $G(x_1, x_2) = \mathcal{G}(x_1, x_2)$? If not, what condition would make the two functions equal?
- For a general $V(\phi)$ obeying the above condition, does $G(x_1, ..., x_4) = 0$?
- Compare $G(x_1, ..., x_4)$ and $\mathcal{G}(x_1, ..., x_4)$.
- How does the free theory differ from the interacting theory in a scattering experiment?

Solutions: 2-point functions

- $$\frac{\delta Z[J]}{\delta J(x_1)} = \frac{\delta}{\delta J(x_1)} e^{iW[J]} = i\frac{\delta W[J]}{\delta J(x_1)} e^{iW[J]}$$

  so

  $$[\frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_1)} = i\frac{\delta^2 W[J]}{\delta J(x_2)\delta J(x_1)} + (i)^2 \frac{\delta W[J]}{\delta J(x_2)} \frac{\delta W[J]}{\delta J(x_1)}] e^{iW[J]}$$

- Evaluate at $J = 0$ and multiply by appropriate powers of i.

  $$\mathcal{G}(x_1, x_2) = G(x_1, x_2) + G(x_2) G(x_1)$$

- Question: When are 1-point functions equal 0? Proposal: Consider Lagrangians where $Z[J] = Z[-J]$.

  $$\frac{\delta Z[J]}{\delta J(x)}|_{J=0} = -\frac{\delta Z[-J]}{\delta J(x)}|_{J=0} = -\frac{\delta Z[J]}{\delta J(x)}|_{J=0}$$

Solutions for even potentials

If $V(\phi) = V(-\phi)$, then $Z[J] = Z[-J]$. Proof:

- Note that $\mathcal{L}(\phi, J) = \mathcal{L}(\phi, 0) + J\phi$.
- $\mathcal{L}(\phi, J) = \mathcal{L}(-\phi, -J)$ since $V(\phi) = V(-\phi)$
- Recall

$$Z[J] = \frac{\int \mathcal{D}\phi e^{i\int d^4x \mathcal{L}(\phi,J)(x)}}{\int \mathcal{D}\phi e^{i\int d^4x \mathcal{L}(\phi,0)(x)}}$$

Change integration variable to $\phi' = -\phi$. Then

$$Z[J] = \frac{\int \mathcal{D}\phi' e^{i\int d^4x \mathcal{L}(-\phi',J)(x)}}{\int \mathcal{D}\phi' e^{i\int d^4x \mathcal{L}(-\phi',0)(x)}} = Z[-J]$$

4-point functions for even potentials

- 3-point functions = 0 with same proof as 1-point functions
- Keep taking derivatives as before and get

$$\mathcal{G}(x_1, x_2, x_3, x_4) = G(x_1, x_2, x_3, x_4) + G(x_1, x_2)G(x_3, x_4) + G(x_1, x_3)G(x_2, x_4) + G(x_1, x_4)G(x_2, x_3)$$

- $G(x_1, x_2, x_3, x_4) = 0$ in a free theory, but not in a general theory.
- In a scattering problem, take Fourier transforms. In the free theory, particles go 'straight through'.

Figure: 4-point functions