Solutions to Exercises for Kachelriess pp 37-40

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Start with a few 1-D identities. Consider a function $f(x)$. Define the Fourier transform $\tilde{f}(k) = \int dxe^{ikx}f(x)$. Then it can be proven [inverse Fourier transform theorem] that $f(x) = \int \frac{dk}{2\pi}e^{-ikx}\tilde{f}(k)$. I'll also use the notation $\mathcal{F}[f](k) \equiv \tilde{f}(k)$.

1. The following identities are commonly used, and you should get used to them. Prove them.

   - $\delta(x) = \int \frac{dk}{2\pi}e^{\pm ikx}$. (Hint: Note that for any function $h(x)$, $\int dxh(x)\delta(x) = h(0)$.)

     Following the hint, we want to show that

     $$\int dxh(x)[\int \frac{dk}{2\pi}e^{\pm ikx}] = h(0)$$

     Change the integration order and use the definition of the Fourier transform of h.

     $$\int dxh(x)[\int \frac{dk}{2\pi}e^{\pm ikx}] = \int \frac{dk}{2\pi}\int dxh(x)e^{\pm ikx} = \int \frac{dk}{2\pi}\tilde{h}(\pm k)$$

     Now observe by changing variables to $k' = \pm k$ that the last integral is $\int \frac{dk'}{2\pi}\tilde{h}(k')$. But by the inverse Fourier transform theorem, this is just $h(0)$ which proves what we set out to show.

   - Let f' be the function defined as $f'(x) = \frac{df}{dx}$. Then $\mathcal{F}[f'](k) = -ik\tilde{f}(k)$.

     From the inverse Fourier transform theorem, write $f(x) = \int \frac{dk}{2\pi}e^{-ikx}\tilde{f}(k)$.

Then take the derivative $\frac{df}{dx}$ so

$$\begin{aligned} f'(x) &= \int \frac{dk}{2\pi} \frac{de^{-ikx}}{dx} \tilde{f}(k) \\ &= \int \frac{dk}{2\pi} e^{-ikx} (-ik\tilde{f}(k)) \end{aligned}$$

This probably would suffice as a proof, but I'd be inclined to go a bit further.

$$\begin{aligned} \mathcal{F}[f'](k) &= \int dx e^{ikx} f'(x) \\ &= \int dx e^{ikx} \int \frac{dk'}{2\pi} e^{-ik'x} (-ik'\tilde{f}(k')) \\ &= \int dk' \int \frac{dx}{2\pi} e^{ix(k-k')} (-ik'\tilde{f}(k')) \\ &= \int \delta(k-k')(-ik'\tilde{f}(k')) \\ &= -ik\tilde{f}(k) \end{aligned}$$

The second to last line employs a delta-function identity very similar to what was demonstrated in the previous exercise (simply substitute x for k and vice versa).

- Let g be the function defined as $g(x) = \sum_{n=1}^{N} a_n \frac{d^n f(x)}{dx^n}$. Then $\mathcal{F}[g](k) = \sum_{n=1}^{N} a_n (-ik)^n \tilde{f}(k)$.

  I'll show this for a quadratic function $g(x) = a_0 + a_1 \frac{df}{dx} + a_2 \frac{d^2 f}{dx^2}$. This should suffice to make the point but if you wanted a formal mathematical proof for general polynomials, you could proceed by induction.

$$\begin{aligned} g'(x) &= \int \frac{dk}{2\pi} [a_0 + a_1 \frac{de^{-ikx}}{dx} + a_2 \frac{d^2 e^{-ikx}}{dx^2}] \tilde{f}(k) \\ &= \int \frac{dk}{2\pi} e^{-ikx} ([a_0 + a_1(-ik) + a_2(-ik)^2] \tilde{f}(k)) \end{aligned}$$

2. Let $\frac{d^2 f(x)}{dx^2} + m^2 f(x) = K(x)$. Find $\tilde{f}(k)$ in terms of $\tilde{K}(k)$.

   Take the Fourier transform of both sides, using above identities.

$$(-ik)^2 \tilde{f}(k) + m^2 \tilde{f}(k) = \tilde{K}(k)$$

so

$$(-k^2+m^2)\tilde{f}(k) = \tilde{K}(k)$$

and

$$\tilde{f}(k) = -\frac{\tilde{K}(k)}{k^2-m^2}$$

3. Let $J(x)$ and $D(x)$ be two functions with Fourier transforms $\tilde{J}(k)$ and $\tilde{D}(k)$. Then define $\mathcal{I}[J] = \int dxdx' J(x)D(x-x')J(x')$. Show that $\mathcal{I}[J] = \int \frac{dk}{2\pi}\tilde{J}^*(k)\tilde{D}(k)\tilde{J}(k)$ where $\tilde{J}^*$ is the conjugate of $\tilde{J}$.

The proof proceeds by substituting, in the $dxdx'$ integrand, the (inverse) Fourier transforms of J and D, then shifting orders of integration and judiciously employing the delta-function identity derived above.

$$\begin{aligned}
&\int dxdx' J(x)D(x-x')J(x') \\
&\quad = \int dxdx' \int \frac{dk}{2\pi} e^{-ikx}\tilde{J}(k) \int \frac{dk'}{2\pi} e^{-ik'(x-x')}\tilde{D}(k') \int \frac{dk''}{2\pi} e^{-ik''x'}\tilde{J}(k'') \\
&\quad = \int \frac{dk}{2\pi} \int \frac{dk''}{2\pi} \int dx e^{-ikx} \int \frac{dk'}{2\pi} \int dx' e^{-ik'(x-x')} e^{-ik''x'}\tilde{J}(k)\tilde{D}(k')\tilde{J}(k'') \\
&\quad = \int \frac{dk}{2\pi} \int \frac{dk''}{2\pi} \int dx e^{-ikx} \int dk' e^{-ik'x} \int \frac{dx'}{2\pi} e^{-ix'(k'-k'')}\tilde{J}(k)\tilde{D}(k')\tilde{J}(k'') \\
&\quad = \int \frac{dk}{2\pi} \int \frac{dk''}{2\pi} \int dx e^{-ikx} \int dk' e^{-ik'x}\delta(k'-k'')\tilde{J}(k)\tilde{D}(k')\tilde{J}(k'') \\
&\quad = \int \frac{dk}{2\pi} \int \frac{dk''}{2\pi} \int dx e^{-ikx} e^{-ik''x}\tilde{J}(k)\tilde{D}(k'')\tilde{J}(k'') \\
&\quad = \int \frac{dk}{2\pi} \int dk'' \int \frac{dx}{2\pi} e^{-ix(k+k'')}\tilde{J}(k)\tilde{D}(k'')\tilde{J}(k'') \\
&\quad = \int \frac{dk}{2\pi} \int dk''\delta(k+k'')\tilde{J}(k)\tilde{D}(k'')\tilde{J}(k'') \\
&\quad = \int \frac{dk}{2\pi}\tilde{J}(k)\tilde{D}(-k)\tilde{J}(-k) \\
&\quad = \int \frac{dk}{2\pi}\tilde{J}(-k)\tilde{D}(k)\tilde{J}(k)
\end{aligned}$$

where the last line comes from changing the integration variable from k to $-k$. Since $J(x)$ is real, $\tilde{J}(-k) = \tilde{J}^*(k)$. This follows from

$$\tilde{J}(-k) = \int dx e^{i(-k)x}J(x) = \int dx[e^{ikx}]^* J(x) = [\int dx e^{ikx}J(x)]^* = \tilde{J}^*(k)$$

Next consider the 2-D Minkowski space with dot product defined by $a \cdot b = a_0 b_0 - a_1 b_1$ for any 2D vectors a and b. Let x and k be 2D vectors. The 2D Fourier transform is given by $\tilde{f}(k) = \int d^2 x e^{ik\cdot x} f(x)$. It can be proven that $f(x) = \int \frac{d^2 k}{(2\pi)^2} e^{-ik\cdot x} \tilde{f}(k)$.

1. Generalize the identities of exercise 1 to 2D. (You needn't prove them if you feel comfortable simply stating them.)

   I'll simply state the generalizations. Let me know if you want help proving them.

   - $\delta(x_1, x_2) = \int \frac{dk_1 dk_2}{(2\pi)^2} e^{ik\cdot x}$
   - $\mathcal{F}[\frac{\partial f(x_1,x_2)}{\partial x_\mu}](k_1, k_2) = -ik_\mu \tilde{f}(k_1, k_2)$ where μ is either 1 or 2.
   - It's a bit messy to write down the most general thing, but the idea is easily illustrated with $g(x_1, x_2) = \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}$. Then $\mathcal{F}[g](k_1, k_2) = (-ik_1)(-ik_2)\tilde{f}(k_1, k_2) = -k_1 k_2 \tilde{f}(k_1, k_2)$.

2. Consider the function $g(x) = \partial_0^2 f(x) - \partial_1^2 f(x) + m^2 f(x)$. Find $\tilde{g}(k)$ in terms of $\tilde{f}(k)$. Suppose for all values of k that $\tilde{f}(k) \neq 0$. Find the values of k so that $\tilde{g}(k) = 0$ and express those values by giving k_0 in terms of k_1 and m. These are the so-called mass-shell values of k.

   Taking the Fourier transform of both sides, we obtain (similarly to what we did in exercise 2 of the 1D case) $\tilde{g}(k_0, k_1) = (-k_0^2 + k_1^2 + m^2)\tilde{f}(k_0, k_1)$. Solve $-k_0^2 + k_1^2 + m^2 = 0$ to get $k_0 = \sqrt{k_1^2 + m^2}$ which is the familiar dispersion equation.