Solutions to Symmetry Exercises

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January 23, 2021

Consider the Pauli matrices (see equation (27) in my *Introduction to Symmetry*)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and also recall the Lie Algebra for rotations (see equation (21) in my *Introduction to Symmetry* or equation (1) in Matthew's *SO3_SU2_notes*)

$$[\mathbf{J_i}, \mathbf{J_j}] = i\epsilon_{ijk}\mathbf{J_k} \tag{2}$$

Also recall (see equation (20) in my *Introduction to Symmetry* and also Matthew's *SO3_SU2_notes*) that a rotation by angle θ around the x-axis can be written as

$$\mathbf{R_x}(\theta) = \exp(i\theta\mathbf{J_1}) \tag{3}$$

where $\mathbf{J_i}$ are matrix representations of the Lie Algebra.

1 Exercise 1

Explicitly, by multiplying the Pauli matrices, verify that they do **NOT** satisfy the Lie Algebra for rotations. That is, show

$$[\sigma_\mathbf{i}, \sigma_\mathbf{j}] \neq i\epsilon_{ijk}\sigma_\mathbf{k}$$

Find 2 x 2 matrices $\tilde{\sigma}_i = \alpha\sigma_i$ where α is a real number, which **DO** satisfy the Lie Algebra.

1.1 Solution

For example:

$$\begin{aligned}\sigma_1\sigma_2 - \sigma_2\sigma_1 &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\\ &= \begin{pmatrix}2i & 0\\ 0 & -2i\end{pmatrix}\\ &= 2i\sigma_3.\end{aligned}$$

So we see that

$$[\sigma_1, \sigma_2] = 2i\sigma_3 \neq i\epsilon^{123}\sigma_3.$$

Pick $\alpha = \frac{1}{2}$ so that $\tilde{\sigma}_i = \frac{1}{2}\sigma_i$.

Then

$$\begin{aligned}\tilde{\sigma}_1\tilde{\sigma}_2 - \tilde{\sigma}_2\tilde{\sigma}_1 &= (\frac{1}{2}\sigma_1)(\frac{1}{2}\sigma_2) - (\frac{1}{2}\sigma_2)(\frac{1}{2}\sigma_1)\\ &= \frac{1}{4}(2i\sigma_3)\\ &= (i\frac{1}{2}\sigma_3) = i\tilde{\sigma}_3 = i\epsilon^{123}\tilde{\sigma}_3.\end{aligned}$$

Similarly for the other indices.

2 Exercise 2a

Expand, equation (3) for a 2D representation, $\mathbf{R_x}(\theta) = \exp(i\theta\tilde{\sigma}_{\mathbf{1}})$, through fourth order in θ.

2.1 Solution

We'll need to compute σ_1^j for $j = 2, 3, 4$. (The cases $j = 1$ and $j = 0$ are trivial.) Since we ultimately want the powers of $\tilde{\sigma}_1$, we'll need to multiply the σ results by the appropriate powers of $\frac{1}{2}$.

$$\sigma_1^2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \sigma_1^3 = \sigma_1, \sigma_1^4 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

For convenience, denote $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now expand $\mathbf{R_x}(\theta) = \exp(i\theta\tilde{\sigma}_\mathbf{1})$.

$$\begin{aligned}
\exp(i\theta\tilde{\sigma}_\mathbf{1}) &= \mathbf{I} + \frac{i\theta}{2}\sigma_1 + \frac{1}{2}i^2\left(\frac{\theta}{2}\right)^2\sigma_1^2 + \frac{1}{3!}i^3\left(\frac{\theta}{2}\right)^3\sigma_1^3 + \frac{1}{4!}i^4\left(\frac{\theta}{2}\right)^4\sigma_1^4 + ... \\
&= \mathbf{I} + \frac{i\theta}{2}\sigma_1 + \frac{1}{2}i^2\left(\frac{\theta}{2}\right)^2\mathbf{I} + \frac{1}{3!}i^3\left(\frac{\theta}{2}\right)^3\sigma_1 + \frac{1}{4!}i^4\left(\frac{\theta}{2}\right)^4\mathbf{I} + ... \\
&= \begin{pmatrix} 1 + \frac{1}{2}i^2(\frac{\theta}{2})^2 + \frac{1}{4!}i^4(\frac{\theta}{2})^4 & \frac{i\theta}{2} + \frac{1}{3!}i^3(\frac{\theta}{2})^3 \\ \frac{i\theta}{2} + \frac{1}{3!}i^3(\frac{\theta}{2})^3 & 1 + \frac{1}{2}i^2(\frac{\theta}{2})^2 + \frac{1}{4!}i^4(\frac{\theta}{2})^4 \end{pmatrix} + ...
\end{aligned}$$

3 Exercise 2b

You will notice in Exercise 2a, that the terms are either proportional to the identity, or else are proportional to $\tilde{\sigma}_1$. (This is an example of applying the Cayley-Hamilton theorem although you don't need to know the theorem to do this exercise.) Extrapolate the results of Exercise 2a to an expression for the n^{th} term of the expansion and find an expression for $\mathbf{R_x}(\theta)$ in terms of trigonometric functions of θ.

3.1 Solutions

$$\exp(i\theta\tilde{\sigma}_\mathbf{1}) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}. \end{pmatrix} \quad (4)$$

4 Exercise 2c

Apply the result of Exercise 2b to a rotation around the x-axis of 360 degrees.

4.1 Solutions

$$\exp(i2\pi\tilde{\sigma}_\mathbf{1}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}. \quad (5)$$