Perturbation Theory Highlights

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For this summary, I will concentrate on a simple toy field theory with the action

$$S[\lambda, J, \phi] = \int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi - \lambda\phi^3) + i\epsilon \tag{1}$$

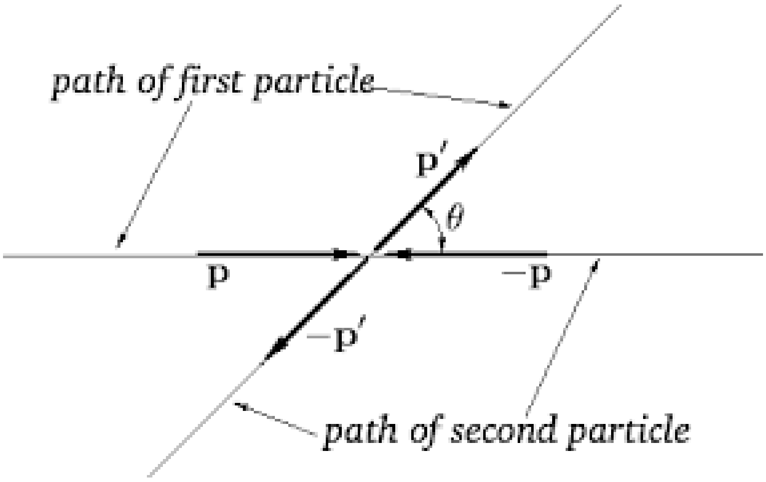
In all expressions below, we will implicitly be taking $\lim_{e\to 0^+}$.

1 Summary 1st Pass

1. Cross-section and scattering $\leftrightarrow$ S-matrix **The S-matrix should not be confused with the action** $S[\lambda, J, \phi]$
2. S-matrix $\leftrightarrow$ Green's functions
3. (starting Kachelreiss) Green's functions $\leftrightarrow$ path integrals
4. Exactly computable free Green's functions $\leftrightarrow$ path integrals for free Lagrangian
5. Full Green's functions $\leftrightarrow$ perturbation expansion of path integrals for full Lagrangian, using free Green's functions
6. Perturbation expansion $\leftrightarrow$ expressed in schematics (Feynman diagrams)
7. (new stuff) Fourier transform $\to$ S-matrix $\to$ experiment.

2 Summary More Details

1. The simplest scattering experiment involves two colliding identical particles.



Ingoing particles

$$p_1^{in} = (E, \vec{p})$$
$$p_2^{in} = (E, -\vec{p})$$

Outgoing particles

$$p_1^{out} = (E', \vec{p}')$$
$$p_2^{out} = (E', -\vec{p}')$$

Question: What is the probability amplitude that if you start with the above ingoing particles as shown, then you'll end up with the above outgoing particles?

Quantum mechanics answer is

$$\langle p_1^{out}, p_2^{out} | S | p_1^{in}, p_2^{in} \rangle$$

where S is the operator propagating *in* to *out* states. S is called the **S matrix**. Its magnitude-squared is also proportional to the (differential) cross section.

2. **LSZ Theorem example.** Assume that the initial and final particles have mass m and are obtained from the quantum field $\phi(x)$. Then (non-trivial proof) !!!

$$\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = (-(p_1^{out})^2 + m^2)(-(p_2^{out})^2 + m^2)(-(p_1^{in})^2 + m^2)(-(p_2^{in})^2 + m^2) [i\int d^4x_1 e^{-ip_1^{in}\cdot x_1}][i\int d^4x_2 e^{-ip_2^{in}\cdot x_2}][i\int d^4x_3 e^{+ip_1^{out}\cdot x_3}][i\int d^4x_4 e^{+ip_2^{out}\cdot x_4}] \mathcal{G}_\lambda(x_1, x_2, x_3, x_4) \tag{2}$$

where $\mathcal{G}_\lambda$ is a Green's function for the toy theory above.

This LSZ example is easily generalized to more ingoing and outgoing particles with multiple masses and associated with other quantum fields.

3.
$$\begin{aligned} G_\lambda(x_1, ..., x_n) &= \frac{1}{Z[\lambda, 0]}\int \mathcal{D}\phi\phi(x_1)...\phi(x_n)e^{iS[\lambda,0,\phi]} \\ &= (-i)^n \frac{1}{Z[\lambda, 0]}\frac{\delta^n Z[\lambda, J]}{\delta J(x_1)...\delta J(x_n)}|_{J(x)=0} \end{aligned} \tag{3}$$

   where
$$Z[\lambda, J] = N\int \mathcal{D}\phi e^{iS[\lambda,J,\phi]} \tag{4}$$

4. From Kachelreiss
$$Z[0, J] = e^{-\frac{1}{2}i\int d^4y d^4y' J(y)\Delta_F(y-y')J(y')} \tag{5}$$

   where
$$\Delta_F(x - x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k^2 - m^2 + i\epsilon} \tag{6}$$

   All free Green functions $G_0(x_1, ..., x_n)$ are obtained by taking derivatives with respect to J **and then setting** $J = 0$.

5. We're ready to compute the Green's function of the full theory where $\lambda \neq 0$. Recall equation (3) and expand the action $S[\lambda, J, \phi]$ from equation (1).

$$\begin{aligned} G_\lambda(x_1, ..., x_n) &= \frac{1}{Z[\lambda, 0]}\int \mathcal{D}\Phi\phi(x_1)...\phi(x_n)e^{i\int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^3) + i\epsilon} \\ &= \frac{1}{Z[\lambda, 0]}\int \mathcal{D}\phi\phi(x_1)...\phi(x_n)e^{iS[0,0,\phi]}e^{-i\lambda\int d^4x\phi^3} \end{aligned} \tag{7}$$

Suppose λ is small, so that we can Taylor-expand the term in red. The Taylor expansion involves products of ϕ's with the exponential of a free ($\lambda = 0$) action. All of these are Green functions G_0 and can be computed analytically

$$\begin{aligned}
&G_\lambda(x_1, ..., x_n) \\
&= \frac{1}{Z[\lambda, 0]} \int \mathcal{D}\phi\phi(x_1)...\phi(x_n)e^{iS[0,0,\phi]}(1 - i\lambda \int d^4x\phi(x)^3 - \frac{\lambda^2}{2} \int d^4x\phi(x)^3 \int d^4x'\phi(x')^3 + ...) \\
&= G_0(x_1, ..., x_n) - i\lambda \int d^4xG_0(x, x, x, x_1, ..., x_n) \\
&\quad - \frac{\lambda^2}{2} \int d^4xd^4x'G_0(x, x, x, x', x', x', x_1, ..., x_n) + ...
\end{aligned} \tag{8}$$

Recall that all free Green's functions are obtained by taking derivatives with respect to J of $Z[0, J]$ given in equation (5) **and then setting** $J = 0$. For example we obtain $G_0(x_1)$ by taking one derivative:

$$\frac{\delta}{\delta J(x_1)}(Z[0, J]) = Z[0, J][-i \int d^4yJ(y)\Delta_F(y - x_1)] \tag{9}$$

6. These derivative expressions rapidly become messy to look at. Fortunately, they can be expressed with schematics known as Feynman diagrams.

$$-i\int d^4y\, J(y)\Delta_F(y-x_1)$$

$$-i\,\Delta_F(x_2-x_1)$$

DIAGRAM RULES

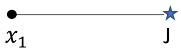
$\frac{\delta}{\delta J_{x_2}}$ Z[0,J] =  Z[0,J]

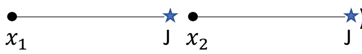
$\frac{\delta}{\delta J_{x_2}}$  = 

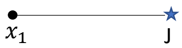
$\frac{\delta}{\delta J_{x_2}}$  = 0

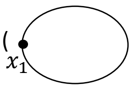
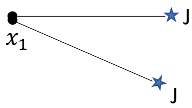
When we set J=0, all lines with ‘J’ are set to 0, and Z is set to Z[0,0]

Examples

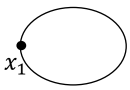
$\frac{\delta}{\delta J_{x_2}}$  Z[0,J]

= ( + ) Z[0,J]

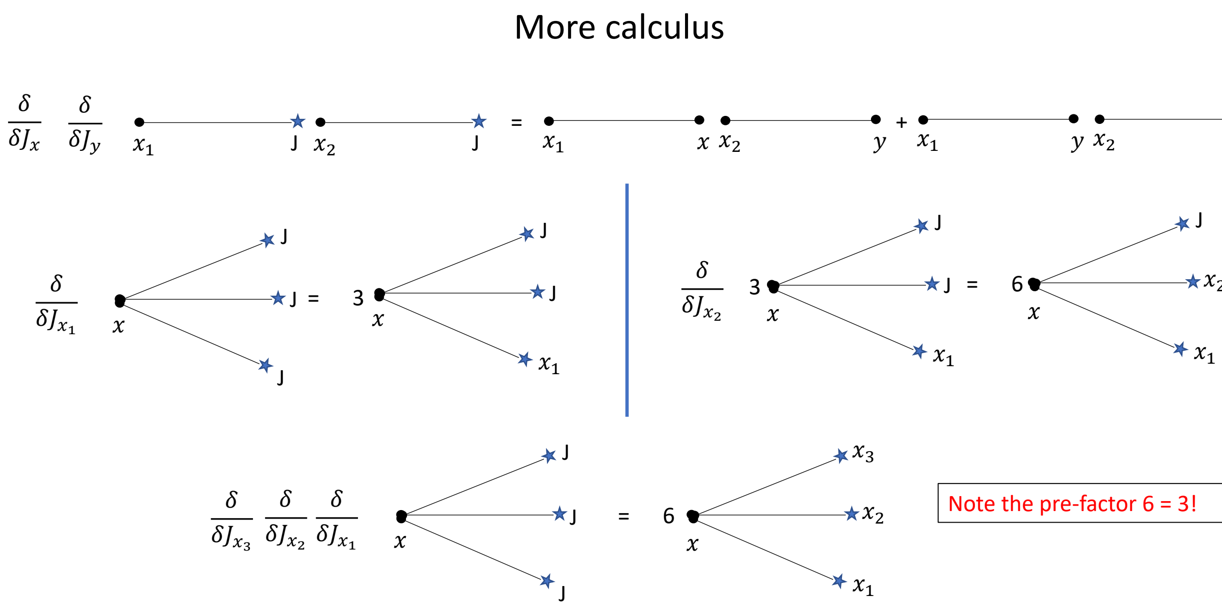
$\frac{\delta}{\delta J_{x_1}}\frac{\delta}{\delta J_{x_1}}$ Z[0,J] = $\frac{\delta}{\delta J_{x_1}}$  Z[0,J]

= ( + ) Z[0,J]

Notice what happens when both J indices are the same

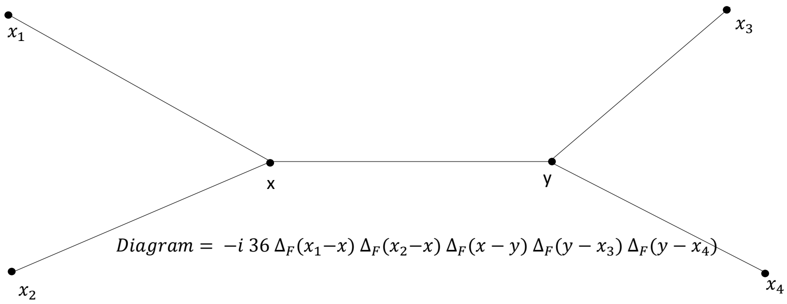
$\frac{\delta}{\delta J_{x_1}}\frac{\delta}{\delta J_{x_1}}$ Z[0,J]$|_{J=0}$ =  Z[0,0]

When we set J=0 only the bubble and Z[0,0] are left



7. Now we're ready to compute an actual S-matrix element (aka cross-section). Recall equation (8) where we showed the perturbation expansion through order λ^2. Apply this to the Green function that appears in the LSZ equation (2). It turns out the only term of interest to us is the one proportional to λ^2 .

$$G_\lambda(x_1, x_2, x_3, x_4) = ... - \frac{\lambda^2}{2} \int d^4x d^4x' G_0(x, x, x, x', x', x', x_1, x_2, x_3, x_4) + ... \tag{10}$$



So $G_\lambda(x_1, x_2, x_3, x_4) = \ldots\, i\frac{\lambda^2}{2} 36 \int d^4x d^4y\, \Delta_F(x_1-x)\, \Delta_F(x_2-x)\, \Delta_F(x-y)\, \Delta_F(y-x_3)\, \Delta_F(y-x_4)$

3 New Stuff

Now we're ready to substitute the Feynman propagator from equation (6) and also to invoke the LSZ theorem of equation (2).

- An important Fourier transform identity is

$$\frac{1}{(2\pi)^4}\int d^4x \int d^4k e^{ipx} e^{-ik(x-y)} f(k) = f(p) e^{ipy} \tag{11}$$

so

$$\begin{aligned}\int d^4x e^{ipx} \Delta_F(x-y) &= \int d^4x e^{ipx}\left(\frac{1}{(2\pi)^4}\right)\int d^4k \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \\ &= \frac{e^{ipy}}{p^2 - m^2 + i\epsilon}\end{aligned} \tag{12}$$

- Recall LSZ

$$\begin{aligned}&\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = \\ &\quad (-(p_1^{out})^2 + m^2)(-(p_2^{out})^2 + m^2)(-(p_1^{in})^2 + m^2)(-(p_2^{in})^2 + m^2) \\ &\quad [i\int d^4x_1 e^{-ip_1^{in}\cdot x_1}][i\int d^4x_2 e^{-ip_2^{in}\cdot x_2}][i\int d^4x_3 e^{+ip_1^{out}\cdot x_3}][i\int d^4x_4 e^{+ip_2^{out}\cdot x_4}] \\ &\quad \mathcal{G}_\lambda(x_1, x_2, x_3, x_4)\end{aligned} \tag{13}$$

Temporarily ignore the terms in blue and compute the remaining terms by using G_λ from the diagram, so the last 3 lines of LSZ are

$$[i\int d^4x_1 e^{-ip_1^{in}\cdot x_1}][i\int d^4x_2 e^{-ip_2^{in}\cdot x_2}][i\int d^4x_3 e^{+ip_1^{out}\cdot x_3}][i\int d^4x_4 e^{+ip_2^{out}\cdot x_4}]\mathcal{G}_\lambda(x_1,x_2,x_3,x_4) =$$
$$\ldots + 18i\lambda^2 \int\int\int\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4x d^4y$$
$$e^{-ip_1^{in}\cdot x_1}\Delta_F(x_1-x)e^{-ip_2^{in}\cdot x_2}\Delta_F(x_2-x)\Delta_F(x-y)e^{+ip_1^{out}\cdot x_3}\Delta_F(y-x_3)e^{+ip_2^{out}\cdot x_4}\Delta_F(y-x_4)$$
$$= \ldots + 18i\lambda^2\int\int d^4x d^4y \frac{e^{-i(p_1^{in}+p_2^{in})\cdot x-(p_1^{out}+p_2^{out})\cdot y)}\Delta_F(x-y)}{(p_1^{in})^2-m^2)((p_2^{in})^2-m^2)((p_1^{out})^2-m^2)((p_2^{out})^2-m^2)} =$$
$$\ldots + 18i\lambda^2(2\pi)^4\frac{\delta(p_1^{in}+p_2^{in}-p_1^{out}-p_2^{out})}{((p_1^{in}+p_2^{in})^2-m^2)((p_1^{in})^2-m^2)((p_2^{in})^2-m^2)((p_1^{out})^2-m^2)((p_2^{out})^2-m^2)} \tag{14}$$

Note that we set the limit $\epsilon \to 0$.

To complete the RHS of the LSZ computation, we must multiply by the terms in blue. But compare these to the last line denominator terms in blue. THEY CANCEL!!!!!! This cancellation is a general feature of LSZ. So finally, using LSZ, we have

$$\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = \ldots + 18i\lambda^2(2\pi)^4\frac{\delta(p_1^{in}+p_2^{in}-p_1^{out}-p_2^{out})}{((p_1^{in}+p_2^{in})^2-m^2)} \tag{15}$$

Notice that the numerator in the last line above is the energy-momentum conserving delta function.

- Recall that for a head-on collision where both particles have the same speed, that

$$\begin{aligned} p_1^{in} &= (E, \vec{p}) \\ p_2^{in} &= (E, -\vec{p}) \end{aligned} \tag{16}$$

Then $(p_1^{in}+p_2^{in})^2 \equiv (p_1^{in}+p_2^{in})\cdot(p_1^{in}+p_2^{in}) = 4E^2$.

Plug this into the S-matrix equation above (equation (15)) to get

$$\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle = \ldots + 18i\lambda^2(2\pi)^4\frac{\delta(p_1^{in}+p_2^{in}-p_1^{out}-p_2^{out})}{4E^2-m^2} \tag{17}$$

- What is this scattering element telling us? This gives us a prediction for what the likelihood of scattering is, depending on the incoming energies. The result depends on 2 parameters λ and m.

- Those parameters have to be measured! So we need at least two measurements at two different energies. After that, we can predict the rest. **If we expanded to higher orders of perturbation theory, we would encounter divergent k integrals.** We deal with this in several steps:
  - First cut off the integral at some large value of k. We typically characterize this value by Λ. Then we make a prediction dependent on λ and m (and also Λ)
  - Once again, we have to determine λ and m from experimental measurements. But this time, the result we get will depend on Λ – an artificial parameter. Now when we make new measurements, we can predict the value for the S-matrix (since the parameters have been determined).
  - Of course, our predictions depend on Λ. But magically, as we make Λ larger and larger (remember, we really want it to be infinite) our predictions converge.
  - And therefore, we have managed to tame the divergences.

  This entire procedure is known as renormalization, and the intermediate step where we decide to cut off the integral is known as a *regularization* method.

- Remember that the mass m is the mass of the particle associated with our field ϕ way back at the beginning in the action given by equation (1). If $2E = m$ then the denominator would be 0 and the S-matrix element would be infinite.

  However, E is the energy of an incoming particle and this energy cannot be less than the particle's rest mass energy m (or in more familiar form, mc^2, when we don't set $c = 1$). As a result we can never have $2E = m$.

  Nevertheless, in more complicated field theories, the incoming particles (fields) can be different than the intermediate particle (i.e., there is a cubic term in the action involving two fields of one type, and one field of another). When that happens, the denominator can indeed become 0 (in practice, there are some higher order perturbative corrections that move things slightly away from 0 so the S-matrix element isn't really infinite).

  Here is what the graph looks like in an actual experiment. The x-axis shows $2E$, otherwise known as the centre-of-mass. The y-axis is proportional to magnitude-squared of the S-matrix, otherwise known as the cross-section. A sharp peak was observed at $2E = 91 GeV$, and this was one of the ways that the Z particle (whose mass is 91 GeV) was discovered.