Symmetry Exercises

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Consider the Pauli matrices (see equation (27) in my *Introduction to Symmetry*)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and also recall the Lie Algebra for rotations (see equation (21) in my *Introduction to Symmetry* or equation (1) in Matthew's *SO3_SU2_notes*)

$$[\mathbf{J_i}, \mathbf{J_j}] = i\epsilon_{ijk}\mathbf{J_k} \tag{2}$$

Also recall (see equation (20) in my *Introduction to Symmetry* and also Matthew's *SO3_SU2_notes*) that a rotation by angle θ around the x-axis can be written as

$$\mathbf{R_x}(\theta) = \exp(i\theta\mathbf{J_1}) \tag{3}$$

where $\mathbf{J_i}$ are matrix representations of the Lie Algebra.

1 Exercise 1

Explicitly, by multiplying the Pauli matrices, verify that they do **NOT** satisfy the Lie Algebra for rotations. That is, show

$$[\sigma_\mathbf{i}, \sigma_\mathbf{j}] \neq i\epsilon_{ijk}\sigma_\mathbf{k}$$

Find 2 x 2 matrices $\tilde{\sigma}_i = \alpha\sigma_i$ where α is a real number, which **DO** satisfy the Lie Algebra.

2 Exercise 2a

Expand, equation (3) for a 2D representation, $\mathbf{R_x}(\theta) = \exp(i\theta\tilde{\sigma}_\mathbf{1})$, through fourth order in θ.

3 Exercise 2b

You will notice in Exercise 2a, that the terms are either proportional to the identity, or else are proportional to $\tilde{\sigma}_1$. (This is an example of applying the Cayley-Hamilton theorem.) Extrapolate the results of Exercise 2a to an expression for the n^{th} term of the expansion and find an expression for $\mathbf{R_x}(\theta)$ in terms of trigonometric functions of θ.

4 Exercise 2c

Apply the result of Exercise 2b to a rotation around the x-axis of 360 degrees.