Introduction to Symmetry – Expanded

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How these notes differ from the Introduction to Symmetry notes from December: The original notes and also these, fall into roughly two parts – (a) generalities about symmetries (b) some specific considerations for quantum mechanics. In these expanded notes, the section on quantum mechanics is preceded by a summary of the key points we'll need later about symmetries. The section after quantum mechanics is also expanded a bit to comment more about the significance of the phases of quantum states.

The subject of symmetries in quantum field theory, is the basis of many important – and different – areas of discovery. We can pick and choose which ones to examine. Some are:

- How to construct Lagrangians with many fields.
- The symmetry structure of quantum electromagnetism (QED)
- Properties of spin particles like the electron – and their equations of motion, the Dirac equation
- Internal symmetries and the origins of families of elementary particles
- Green function identities for scattering of spin 1/2 particles

1 Outline

- Definition and examples of symmetries
- Representations of 2D rotations – classical
- Rotations in 3 dimensions
- Review of quantum mechanics
- Representations of 3D rotations – quantum

2 Definition and examples of symmetries

A symmetry is an operation on field and space-time variables, which doesn't change the laws (aka equations) of physics.

Comment: Not all symmetries are useful. Usefulness has to do with the implications for how particle states are related and how S-matrix elements are related.

- **Example 0: The Most Important Symmetry** Consider the equation of motion

  $$[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}]\phi(t,x,y) + m\phi(t,x,y) = 0$$

  Operate on the time variable by **time-translation**

  $$t = t' + \alpha$$

  and on the field by

  $$\phi'(t', x, y) = \phi(t, x, y)$$

  Since $\frac{\partial\phi(t,x,y)}{\partial t} = \frac{\partial\phi'(t',x,y)}{\partial t'}$, the equation of motion isn't changed so time-translation is a symmetry.

- **Example 1:**

  Again, consider the equation of motion

  $$[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}]\phi(t,x,y) + m\phi(t,x,y) = 0$$

  A symmetry is

  $$\phi(t,x,y) \to -\phi(t,x,y)$$

  because this doesn't change the equation.

- **Example 2:**

  $$[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}]\phi(t,x,y) + m\phi^2(t,x,y) = 0$$

  $$\phi(t,x,y) \to -\phi(t,x,y)$$

  is **NOT** a symmetry because this does change the equation.

- **Example 3:**

  Same equation as example 1.

$$[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}]\phi(t,x,y) + m\phi(t,x,y) = 0 \tag{1}$$

  Consider the operation (aka 'transformation'), representing a rotation of angle θ. Abstractly describe the operation as $\mathcal{O}(\theta)$.

  WARNING!!! I goofed here. Usually the angle θ is chosen to represent a counter-clockwise rotation. However, with the definitions that follow, θ has been chosen to be a clockwise rotation. With that understanding, the rest of these notes regarding 2D, are self-consistent. However, for people accustomed to the counter-clockwise convention, the rotation shown here appears to be a "passive rotation" – i.e, one in which the coordinate systems are transformed rather than the x, y vector. And likewise, the transformation shown later for the 2D fields, appears (in the counter-clockwise convention) to be an "active rotation". (Thanks for Mahendra for pointing this out.)

$$\begin{aligned} t' &= t \\ x'(x,y) &= x\cos\theta + y\sin\theta \\ y'(x,y) &= -x\sin\theta + y\cos\theta \\ \phi'(t',x',y') &= \phi(t,x,y) \end{aligned} \tag{2}$$

  We will show this **IS** a symmetry of the equation because it <u>does not change the equation</u>.

  Apply the chain rule.

$$\begin{aligned} \frac{\partial\phi(t,x,y)}{\partial x} &= \frac{\partial\phi'(t',x'(x,y),y'(x,y))}{\partial x} \\ &= \frac{\partial\phi'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial\phi'}{\partial y'}\frac{\partial y'}{\partial x} \\ &= \frac{\partial\phi'}{\partial x'}\cos\theta - \frac{\partial\phi'}{\partial y'}\sin\theta \end{aligned} \tag{3}$$

  Then apply it again.

$$\frac{\partial^2\phi(t,x,y)}{\partial x^2} = -2\frac{\partial^2\phi'}{\partial x'\partial y'}\cos\theta\sin\theta + \frac{\partial^2\phi'}{\partial x'^2}\cos^2\theta + \frac{\partial^2\phi'}{\partial y'^2}\sin^2\theta \tag{4}$$

Similarly

$$\frac{\partial^2\phi(t,x,y)}{\partial y^2} = 2\frac{\partial^2\phi'}{\partial x'\partial y'}\cos\theta\sin\theta + \frac{\partial^2\phi'}{\partial x'^2}\sin^2\theta + \frac{\partial^2\phi'}{\partial y'^2}\cos^2\theta \tag{5}$$

Noting that $\cos^2\theta + \sin^2\theta = 1$, add the two equations to get

$$\frac{\partial^2\phi(t,x,y)}{\partial x^2} + \frac{\partial^2\phi(t,x,y)}{\partial y^2} = \frac{\partial^2\phi'(t',x',y')}{\partial x'^2} + \frac{\partial^2\phi(t',x',y')}{\partial y'^2} \tag{6}$$

and also notice that

$$m\phi(t,x,y) = m\phi'(t',x',y') \tag{7}$$

Compare these to equation (1), proving symmetry.

- **Example 4:**

  Take the same equation as Examples 1 and 3.

$$[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}]\phi(t,x,y) + m\phi(t,x,y) = 0 \tag{8}$$

  Consider the operation representing a Lorentz boost.

$$\begin{aligned} y' &= y \\ x'(t,x) &= \gamma(v)(x+vt) \\ t'(t,x) &= \gamma(v)(t+vx) \\ \phi'(t',x',y') &= \phi(t,x,y) \end{aligned} \tag{9}$$

  where $\gamma(v) = (\sqrt{1-v^2})^{-1}$.

  We can show this is a symmetry.

3 Representations of 2D Rotations – Classical

3.1 One field – again

$$[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}]\phi(t,x,y) + m\phi(t,x,y) = 0 \tag{10}$$

Recall the operation (aka 'transformation'), $\mathcal{O}(\theta)$, for a rotation of angle θ.

$$\begin{aligned} t' &= t \\ x'(x,y) &= x\cos\theta + y\sin\theta \\ y'(x,y) &= -x\sin\theta + y\cos\theta \\ \phi'(t',x',y') &= \phi(t,x,y) \end{aligned} \tag{11}$$

Notice a trivial but important thing. If you first rotate by θ_1 and then by θ_2, we can apply the equations twice to show that the net transformation is equivalent to a rotation of $\theta_1 + \theta_2$. (We'll prove this later.) We write this abstractly as $\mathcal{O}(\theta_2)\mathcal{O}(\theta_1) = \mathcal{O}(\theta_2+\theta_1)$. **We say that the family of operators $\mathcal{O}(\theta)$ form a REPRESENTATION OF THE 2D ROTATION GROUP.** This terminology will be explained in the next section.

3.2 Two fields

Suppose there are two fields, ϕ_1 and ϕ_2. Change the operation $\mathcal{O}(\theta)$ of equation (2) to

$$\begin{aligned} t' &= t \\ x'(x,y) &= x\cos\theta + y\sin\theta \\ y'(x,y) &= -x\sin\theta + y\cos\theta \\ \phi_1'(t',x',y') &= \cos(\theta)\phi_1(t,x,y) - \sin(\theta)\phi_2(t,x,y) \\ \phi_2'(t',x',y') &= \sin(\theta)\phi_1(t,x,y) + \cos(\theta)\phi_2(t,x,y) \end{aligned} \tag{12}$$

We have not yet shown equations of motion for which the above transformation is a symmetry. (It turns out that it's a bit nontrivial to construct equations that exhibit these symmetries.)

Written in matrix form the equations in red are:

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \equiv \mathbf{R}(\theta) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{13}$$

where the matrix $\mathbf{R}(\theta)$ has been introduced.

Suppose you rotate twice, first by θ_1 and then by θ_2, so

$$\begin{pmatrix} \phi_1'' \\ \phi_2'' \end{pmatrix} = \mathbf{R}(\theta_2)\mathbf{R}(\theta_1) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{14}$$

Compute the matrix product. For example, the first column, first row:

$$\cos(\theta_2)\cos(\theta_1) - \sin(\theta_2)\sin(\theta_1) = \cos(\theta_2+\theta_1) \tag{15}$$

When you compute all 4 elements, you find $\mathbf{R}(\theta_2)\mathbf{R}(\theta_1) = \mathbf{R}(\theta_2+\theta_1)$. **This is not a coincidence! It is a requirement**.

3.2.1 Groups and representations

Symmetry operations come in families known as **groups**. A group is a set G together with an binary operation on G, often denoted $\cdot$, that combines any two elements a and b to form another element of G, denoted $a \cdot b$, in such a way that the following three requirements, known as group axioms, are satisfied:

- Associativity For all a, b, c in G, one has $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- Identity element There exists an element e in G such that, for every a in G, one has $e \cdot a = a$ and $a \cdot e = a$. Such an element is unique. It is called the identity element of the group.
- Inverse element For each a in G, there exists an element b in G such that $a \cdot b = e$ and $b \cdot a = e$, where e is the identity element. For each a, the element b is unique; it is called the inverse of a and is commonly denoted a^{-1}.

In the example of 2D rotations, we can identify the group elements with the rotation angle θ modulo 2π (i.e. angles are equivalent to each other when they differ by 360 degrees) and the binary operation as "the sum of two angles" e.g. $\theta_1 + \theta_2$. The identity is 0 and the inverse of θ is $-\theta$.

A (linear) **representation** (when we use the term in physics) is a mapping from the group G to linear operators (for example matrices) which are denoted as $\mathcal{O}(a)$ with the properties

- $\mathcal{O}(a)\mathcal{O}(b) = \mathcal{O}(a \cdot b)$
- $\mathcal{O}(e) = \mathbf{I}$
- $\mathcal{O}(a^{-1}) = \mathcal{O}^{-1}(a)$ [This isn't actually an independent property.]

So, for example, for 2D rotations, we might try to find a set of 2 x 2 matrices $\mathbf{R}(\theta)$ so that $\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2)$. Importantly, $\mathbf{R}(2\pi) = \mathbf{I}$. If we can find such a set of matrices, we would call this a 2D representation of the group of rotations.

3.2.2 Lie Groups

A **Lie Group** is an (infinite) group which is also a differentiable manifold. It's usually sufficient to think of this as some kind of a surface that can locally be 'mapped' to a a finite set of parameters like θ. (The surface of a sphere is a manifold.)

The group of 2D rotations is sometimes known as $O(2)$ and is an example of a Lie Group.

3.2.3 Reducible and irreducible representations

For now, I don't think it's important to understand in any detail what is meant by the terms 'reducible' and 'irreducible'. Roughly speaking, if you think of representation $\mathcal{O}(a)$ as matrices, then the representation is **reducible** if it's possible to write all of the matrices in a specific block form, for example

$$\mathcal{O}(a) = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{16}$$

If such a decomposition can't be found, then the representation is called **irreducible**. Sometimes, reducible representations are considered trivial since you can easily build them out of smaller blocks.

Physicists usually analyze irreducible representations, since reducible representations can be "factored" into irreducible representations.

4 Rotations in 3 Dimensions

4.1 Constructing the Lie group

Rotations around the x, y and z axes are:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}, \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix}, \begin{pmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

which we will write as

$$\mathbf{R_x}(\theta_\mathbf{x})\,, \mathbf{R_y}(\theta_\mathbf{y})\,, \mathbf{R_z}(\theta_\mathbf{z}) \tag{18}$$

Any 3D rotation can be written as a product of these 3 rotations for some choice of angles. Define $\hat{\mathbf{R}}(\theta_\mathbf{x}, \theta_\mathbf{y}, \theta_\mathbf{z}) \equiv \mathbf{R_x}(\theta_\mathbf{x})\mathbf{R_y}(\theta_\mathbf{y})\mathbf{R_z}(\theta_\mathbf{z})$. A more common description of 3D rotations is expressed as $\mathbf{R}(\boldsymbol{\theta}) \equiv \hat{\mathbf{R}}(\theta_\mathbf{x}, \theta_\mathbf{y}, \theta_\mathbf{z})$ where $\boldsymbol{\theta}$ is defined by noting that the rotation $\hat{\mathbf{R}}(\theta_\mathbf{x}, \theta_\mathbf{y}, \theta_\mathbf{z})$ is equivalent to a rotation of an angle $|\boldsymbol{\phi}|$ around an axis whose direction is the same as that of the 3-vector $\boldsymbol{\phi}$, where $\boldsymbol{\phi}$ is to be determined (and whose existence is guaranteed by Euler's theorem – see notes emailed by Mahendra).

4.2 The Lie Algebra

Consider $\mathbf{R_x}(\theta_\mathbf{x})$. Expand it as a Taylor expansion for small values of θ_x.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + i\theta_x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} + ... \tag{19}$$

Define $\mathbf{I} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{J_x} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$.

We can rewrite Equation (19) as $\mathbf{R_x}(\theta_\mathbf{x}) = \mathbf{I} + i\theta_x \mathbf{J_x} + ...$ It turns out that if we continue the Taylor expansion, we find

$$\mathbf{R_x}(\theta_\mathbf{x}) = \mathbf{I} + i\theta_x \mathbf{J_x} + \frac{(i\theta_x^2)}{2}\mathbf{J_x}^2 + ... = \exp(i\theta_x \mathbf{J_x}) \tag{20}$$

We've shown is that all rotations θ_x around the x-axis, can be obtained by the exponential of $i\theta_x \mathbf{J_x}$. We say that $\mathbf{J_x}$ *generates* all rotations around the x-axis and $\mathbf{J_x}$ is called the **infinitesimal generator of rotations around the x-axis**. Similarly, we can find the infinitesimal generators $\mathbf{J_y}$ and $\mathbf{J_z}$.

Since all rotations can be obtained by a sequence of rotations around the x, y and z axes, we have now characterized all rotations in terms of the infinitesimal generators. Notice that if you rotate by θ_x around the x-axis, and then by θ_y around the y-axis, that is not the same as rotating first by θ_y around the y-axis and then by θ_x around the x-axis. That is,

$$\mathbf{R_y}(\theta_\mathbf{y})\mathbf{R_x}(\theta_\mathbf{x}) \neq \mathbf{R_x}(\theta_\mathbf{x})\mathbf{R_y}(\theta_\mathbf{y})$$

The operations don't commute. (This feature is absent in 2D.)

It follows that for the infinitesimal generators

$$\mathbf{J_y}\mathbf{J_x} \neq \mathbf{J_x}\mathbf{J_y}$$

We can directly verify this with the matrix expressions for $\mathbf{J_x}$ (which was derived above) and $\mathbf{J_y}$ (not derived above – exercise for reader). We can rewrite the above inequality as

$$[\mathbf{J_y}, \mathbf{J_x}] \neq 0$$

where we use the **Lie bracket** notation $[\mathbf{J_y}, \mathbf{J_x}] \equiv \mathbf{J_y}\mathbf{J_x} - \mathbf{J_x}\mathbf{J_y}$.

Furthermore, by deriving the matrix expressions for $\mathbf{J_x}$ (done above), $\mathbf{J_y}$ and $\mathbf{J_z}$ we can show

$$[\mathbf{J_i}, \mathbf{J_j}] = i\epsilon_{ijk}\mathbf{J_k} \tag{21}$$

where we've equated the indices $(1, 2, 3)$ with (x, y, z).

The generators J_i together with the commutation relations of equation (21) are known as the **LIE ALGEBRA of the rotation group.** $\mathbf{J_i}$ are called angular momentum operators.

Every group has a unique Lie Algebra. When people want to irreducible representations, the usual method is to start with the Lie Algebra.

4.3 The Hamiltonian

Remember our first symmetry example, time-translation. $t \to t + \alpha$ is a symmetry for any value of α. Write the time-translation operator as $\hat{\mathbf{T}}(\alpha)$.

Call the infinitesimal generator $\mathbf{H}$ so that $\hat{\mathbf{T}}(\alpha) = \exp(i\alpha\mathbf{H})$. $\mathbf{H}$ is called the Hamiltonian.

Notice that time-translations are completely independent of rotations. We express this by the Lie Algebra rule

$$[\mathbf{H}, \mathbf{J_i}] = 0 \tag{22}$$

That is, the Hamiltonian commutes with the angular momentum operators (rotation generators). The Hamiltonian commutes with most symmetry operations. That is one of the most important facts about quantum mechanics and quantum field theory.

4.4 Representations

What kind of representations can be constructed? Specifically, we are interested in families of $N \times N$ matrices $U(\theta_x, \theta_y, \theta_z)$ representing the rotations $\hat{\mathbf{R}}(\theta_\mathbf{x}, \theta_\mathbf{y}, \theta_\mathbf{z})$ so that if $\hat{\mathbf{R}}(\theta_\mathbf{x}, \theta_\mathbf{y}, \theta_\mathbf{z}) = \hat{\mathbf{R}}(\theta^\mathbf{1}_\mathbf{x}, \theta^\mathbf{1}_\mathbf{y}, \theta^\mathbf{1}_\mathbf{z})\hat{\mathbf{R}}(\theta^\mathbf{2}_\mathbf{x}, \theta^\mathbf{2}_\mathbf{y}, \theta^\mathbf{2}_\mathbf{z})$ then $U(\theta_x, \theta_y, \theta_z) = U(\theta^1_x, \theta^1_y, \theta^1_z)U(\theta^2_x, \theta^2_y, \theta^2_z)$. We already know an example for 3×3 matrices – namely in that case set $U = \hat{\mathbf{R}}$. But what if $N \neq 3$?.

One example has $N = 5$. This representation occurs, for example, when characterizing the electrons in an atomic d-shell (which has 5 electrons of each spin). We write U operating on the fields $\mathbf{\Phi}$ in the form

$$U(\theta_x, \theta_y, \theta_z)\mathbf{\Phi} \equiv \begin{pmatrix} a^{11} & a^{12} & a^{13} & a^{14} & a^{15} \\ a^{21} & a^{22} & a^{23} & a^{24} & a^{25} \\ a^{31} & a^{32} & a^{33} & a^{34} & a^{35} \\ a^{41} & a^{42} & a^{43} & a^{44} & a^{45} \\ a^{51} & a^{52} & a^{53} & a^{54} & a^{55} \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \\ \phi^5 \end{pmatrix} \tag{23}$$

where the coefficients are each derivable functions of the three basic angles and furthermore, no matrix transformation can convert the above into block form.

It turns out there are irreducible representations for all odd integers.

5 Summary and significance of the key facts about symmetries

- A symmetry is an operation on field and space-time variables, which doesn't change the laws (aka equations) of physics.
- The study of symmetries in physics consists mostly of two areas:
  - What kinds of operations can be performed on the fields and space-time variables. We call this *representation theory.*
  - What combinations of fields and operators (making up the "action"), when operated-upon by symmetry-representations, are invariant under symmetry transformations? This is a fancy way of saying "how can we write equations that don't change the laws of physics?" Another fancy way of saying this is "what combinations of fields and operators can result in scalars?" **This is NOT something covered in these introductory notes.**
- **Group representations**: The symmetry operations must preserve the essential properties of the symmetries. For example, if you rotate around the y-axis by 90 degrees and then around the x-axis by 90 degrees, that is the same as rotating around the axis $\begin{pmatrix}-1 & -1 & 1\end{pmatrix}$ by 60 degrees. Thus the operation (representation) for the first rotation followed by the operation for the second, should be the same as the operation for rotation around the axis $\begin{pmatrix}-1 & -1 & 1\end{pmatrix}$ by 60 degrees.
- **Representation of 360 degree rotation**: In particular, the operation for rotation by 360 degrees **must be the same as doing nothing!** (The identity.)
- Rotations can be represented as N x N matrices operating on N-dimensional vector spaces. **These are called finite-dimensional linear representations of the rotation group, also known as SO(3).**
- The mathematical methodology for classifying representations of the rotation group, is to start with the much easier problem of constructing

representations of the **Rotation Lie Algebra. The generators of that Lie Algebra are denoted J_i.**

- **If there is an NxN representation of rotations**, then if $\tilde{\mathbf{J}}_i$ are the Lie Algebra generators in that representation, then rotations are represented as $e^{i\tilde{\mathbf{J}}\cdot\theta}$.
- **BUT** it turns out that the **rotation group only has odd-dimension representations** (N is odd) **although** there are even-dimension representations of the rotation Lie Algebra!
  - For those, $e^{i\tilde{\mathbf{J}}\cdot\theta}$ represent elements of the group $SU(2)$ **which is NOT the rotation group!**

6 Review of Quantum Mechanics

We will work in the Heisenberg picture, where operators are time-dependent and states are time-independent.

- A state is a vector in a Hilbert space and is often written as $|\psi\rangle$.
- A superposition of states is represented by a sum of states, e.g. $\alpha|\psi\rangle + \beta|\gamma\rangle$.
- The inner product between state $|\psi\rangle$ and state $|\gamma\rangle$ is written $\langle\gamma|\psi\rangle$.
  - The left-hand state is complex-conjugated. That is, if $|\tilde{\psi}\rangle = e^{i\alpha}|\psi\rangle$ and $|\tilde{\gamma}\rangle = e^{i\beta}|\gamma\rangle$, then $\langle\tilde{\gamma}|\tilde{\psi}\rangle = e^{i(\alpha-\beta)}\langle\gamma|\psi\rangle$.
  - The probability of state $|\gamma\rangle$ being observed given that we know we are in state $|\psi\rangle$ is $\frac{|\langle\gamma|\psi\rangle|^2}{\langle\psi|\psi\rangle\langle\gamma|\gamma\rangle}$.
  - Thus the state $|\psi\rangle$ cannot be physically distinguished from $|\tilde{\psi}\rangle$.
  - When $\langle\psi|\psi\rangle = 1$ we say that the state $|\psi\rangle$ is <u>normalized</u>.
  - When $\langle\psi|\gamma\rangle = 0$ we say that the states $|\psi\rangle$ and $|\gamma\rangle$ are <u>orthogonal</u> to one another. When those states are also normalized, we say that the two states are <u>orthonormal</u> to one another.
- An operator acts on the state by changing it to another state. E.g. $|\psi'\rangle = O|\psi\rangle$. In this expression, O is the operator.

  We always consider **linear operators**. Linearity is defined by $O(\alpha|\psi\rangle + \beta|\gamma\rangle) = \alpha O|\psi\rangle + \beta O|\gamma\rangle$.

- Measurable quantities are represented by self-adjoint operators ($O^\dagger = O$).

- An eigenstate of O is a state obeying $O|\psi\rangle = \lambda|\psi\rangle$. In this equation, λ is called an eigenvalue.

- In general, measurement operators O depend on time as $O(t)$. This is the Heisenberg picture.

- The average value, in the state $|\psi\rangle$, of the measurement represented by the operator O is $\langle\psi|O|\psi\rangle$.

- In Quantum Mechanics, we start with a space of states and then construct the operators corresponding to measurements. In QFT, we start with the fields – which are operators – and derive from their commutation relations what the space of states must be! That's how we deduce that the states represent properties of particles.

- An operator U is unitary if $U^\dagger = U^{-1}$.

  - If $|\tilde{\psi}\rangle = U|\psi\rangle$ and $|\tilde{\gamma}\rangle = U|\gamma\rangle$, then $\langle\tilde{\gamma}|\tilde{\psi}\rangle = \langle\gamma|\psi\rangle$.

  We see that unitary operators preserve probabilities.

- A symmetry is an operation which doesn't change the laws of physics. So a symmetry must be an operator that preserves probabilities.

  Symmetries are represented by unitary operators.

- It often happens that there is a **family of operators**. Two examples:

  - $\phi(t, x, y)$ describes a field operator acting on a 2-D space, and varying with time.
  - $U(\theta)$ can be used to describe a family of operators parametrized by θ and representing rotations in 2 dimensions.

- Sometimes $U(\theta) = \exp(i\theta L) \equiv I + i\theta L - \frac{1}{2}\theta^2 L^2 + \ldots$ where L is a self-adjoint operator (thus a measurement operator).

  Then we only need to think about one operator L instead of a whole family of operators.

7 Representations of 3D rotations – quantum

7.1 Equations of motion and symmetry operators

In field theory, the equations of motion are the Euler-Lagrange equations for the fields of the theory. In quantum field theory, the fields are measurable (self-adjoint) operators.

As explained in the section on quantum mechanics – a symmetry transformation representing the rotation R, is represented by a unitary operator, $U(R)$, acting on the space of states, e.g. $U(R)|\psi\rangle$. The operation of $U(R)$ on a collection of fields represented by the N-vector $\mathbf{\Phi} = (\phi^1, ..., \phi^N)$, is $U\mathbf{\Phi}U^\dagger$. So we describe transformed objects as

$$\begin{aligned} |\psi'\rangle &= U|\psi > \\ \mathbf{\Phi}'(t', \mathbf{x}') &= U\mathbf{\Phi}(t, \mathbf{x})U^\dagger \end{aligned} \tag{24}$$

These transformations are symmetries if the transformed fields and space-time parameters obey the same equations of motion.

7.2 Representation up to a phase

Remember that $|\psi\rangle$ is the same physical state as $-|\psi\rangle$. So, if $U(\theta_z)$ is a unitary operator representing a rotation by θ_z around the z-axis, then it's OK if $U(2\pi)|\psi\rangle = -|\psi\rangle$.

It is permissible to represent the rotation group with a family of operators that obey the same Lie Algebra equations as the rotation Lie Algebra, but where, for example, $U(\theta_z + 2\pi) = -U(\theta_z)$. In other words, in QM the representations of interest are representations of SU(2). This is an important difference from classical mechanics.

Note that the issue of phase is not as trivial as it sounds. For example, even though $|\psi\rangle$ is the same as $-|\psi\rangle$, and $|\phi\rangle$ is the same as $-|\phi\rangle$, the state $|\psi\rangle + |\phi\rangle$ is not the same as the state $|\psi\rangle - |\phi\rangle$. One of the profound facts about particle states that transform as even-dimensional representations (i.e. where a 360-degree 'rotation' changes sign by -1), is that there cannot be 2 particles in identical states. This is the famous Pauli exclusion principle.

7.3 Degeneracy of energy eigenvalues

Since rotations are independent of time-translation, we know that $[\mathbf{H}, \mathbf{J}_i] = 0$. Let $|E, k\rangle$ be an eigenstate of $\mathbf{H}$ with eigenvalue E. The index k is a placeholder for the possibility that there may be multiple orthogonal eigenstates of $\mathbf{H}$ with the same eigenvalue. So, we have $\mathbf{H}|E,k\rangle = E|E,k\rangle$. **If there is more than one such state (i.e., more than one value of k) orthogonal to one another, then we say that the energy is degenerate.**

From the commutation relationship between the Hamiltonian $\mathbf{H}$ and angular momentum operator $\mathbf{J}_z$

$$\mathbf{H}\mathbf{J}_z|E,k\rangle = \mathbf{J}_z\mathbf{H}|E,k\rangle = E\mathbf{J}_z|E,k\rangle \tag{25}$$

What we've shown is that

$$\mathbf{H}[\mathbf{J}_z|E,k\rangle] = E[\mathbf{J}_z|E,k\rangle] \tag{26}$$

meaning that $[\mathbf{J}_z|E,k\rangle]$ is an eigenstate of the Hamiltonian, with eigenvalue E.

Consider two possibilities. The first possibility is that the state $|E,k\rangle$ happens also to be an eigenstate of $\mathbf{J}_z$ so $\mathbf{J}_z|E,k\rangle = j_z|E,k\rangle$ for some value j_z. In that case, $\mathbf{J}_z|E,k\rangle$ is proportional to $|E,k\rangle$ and thus **not** orthogonal. So we cannot conclude from this, whether or not the energy eigenvalue is degenerate.

The second possibility is that the state $|E,k\rangle$ is not an eigenstate of $\mathbf{J}_z$. In that case $\mathbf{J}_z|E,k\rangle$ is not proportional to $|E,k\rangle$ so there are at least two orthogonal vectors with the same energy eigenvalue (it's a simple result of linear algebra that if there exists two non-orthogonal vectors, you can always find linear combinations of them which are orthogonal). So then the energy eigenvalue is degenerate.

More generally, suppose that rotations are represented as N-dimensional matrices (i.e., an N-dimensional representation) acting on states. Then the above arguments show that the energy eigenvalues are all (at least) N-fold degenerate. This is the origin of the fact that shells in the periodic table have certain numbers of electrons all of approximately the same energy, and that elementary particles can be found in polarization multiplets of equal mass (*mass = energy!*).

7.4 Spin 1/2

In the classical theory, we discussed 2 x 2 representations of the 2D rotations. We also said (or at least implied) that for 3D rotations, there were no 2 x 2 irreducible representations. However, we only were looking at **exact** representations and not **representations up to a phase**. Here is a 2 x 2 irreducible representation – up to a phase – of the 3D rotation group.

First define the 2 x 2 Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{27}$$

and write $\boldsymbol{\sigma}$ as the 3-tuple $(\sigma_1, \sigma_2, \sigma_3)$.

Then the 2 x 2 matrix representing (up to a phase) a rotation of angle $\theta \equiv |\boldsymbol{\theta}|$ around an axis in the direction of the 3-vector $\boldsymbol{\theta}$ is

$$\mathbf{R}(\boldsymbol{\theta}) = e^{i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta}} = \begin{pmatrix} \cos\frac{\theta}{2} + i\hat{\theta}_3 \sin\frac{\theta}{2} & (i\hat{\theta}_1 + \hat{\theta}_2)\sin\frac{\theta}{2} \\ (i\hat{\theta}_1 - \hat{\theta}_2)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} - i\hat{\theta}_3 \sin\frac{\theta}{2} \end{pmatrix} \tag{28}$$

where the notation $\hat{\theta}_i$ denotes $\frac{\theta_i}{\theta}$.

It's easy to see that when $\tilde{\theta} = 2\pi$, we have $\mathbf{R}(\tilde{\theta}) = -\mathbf{I}$.

Why do we call this " spin 1/2"? The definition of spin is this: Let J_z be represented by an N-dimensional matrix. Then find the eigenvalues of that matrix. The maximum eigenvalue, j, is called the spin. It can be shown that $j = \frac{N-1}{2}$. So if $N = 2$, then $j = \frac{1}{2}$. Similarly, for 3-dimensional representations, the spin is 1.