Remarks on Symmetry Introduction

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During our meeting on Monday December 28, there were several questions that came up which I noted. Perhaps there are more I've forgotten about, but in the meantime, here are some responses. Others of you may have things you want to add.

1 Is equation 12 correct?

To remind you, equation (12) is the transformation

$$\begin{aligned} t' &= t \\ x'(x,y) &= x\cos\theta + y\sin\theta \\ y'(x,y) &= -x\sin\theta + y\cos\theta \\ \phi'_1(t',x',y') &= \cos(\theta)\phi_1(t,x,y) - \sin(\theta)\phi_2(t,x,y) \\ \phi'_2(t',x',y') &= \sin(\theta)\phi_1(t,x,y) + \cos(\theta)\phi_2(t,x,y) \end{aligned}$$

The question is whether, on the RHS of the red lines, the arguments are really t, x, y or should they be t', x', y'?

The answer is that the equation is correct **as is**. Namely, the arguments really are t, x, y. I believe that José explained this during our meeting, but for those who want to see it written down, look at Lancaster equation 9.46. The form of the equation looks a bit different but the point is that the argument(s) of ϕ on the LHS, are related to those on the RHS by the transformation $\mathbf{x} = \mathbf{R}^{-1}(\theta)\mathbf{x}'$.

2 How is the stuff we learned in QM about angular momentum related to the stuff we are learning about field transformations?

First of all, let's concentrate in QM on wave functions of the form

$$\psi(x, y, z).$$

(Elsewhere I use the symbol $|\psi\rangle$ but too much information is missing with that notation.) These wave functions form a Hilbert space under the usual addition operations and with an inner product between the functions $\psi(x, y, z)$ and $\gamma(x, y, z)$ of $\int d^3x\psi^*(\mathbf{x})\gamma(\mathbf{x})$.

Then rotations are represented by operations on the wavefunction and infinitesimal rotations can be expanded in terms of infinitesimal generators. The Hilbert space of wavefunctions is infinite-dimensional, and the transformations are also. For now, consider the infinitesimal generator $\mathbf{J_z}$ which generates rotations around the z-axis. Mostly physicists are lazy with notation, so in the context of wavefunctions we also write the *representation* of $\mathbf{J_z}$ as $\mathbf{J_z}$. To be concrete, the transformation rule using this notation is

$$(\mathbf{J_z}\psi)(x, y, z) = -i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})\psi(x, y, z) \tag{1}$$

where as usual I suppress the Planck constant $\hbar$ and the rotations are around the origin $\mathbf{x} = 0$.

By convention, since the wavefunction in this example has only one component, people use the symbol $\mathbf{L_z}$ rather than $\mathbf{J_z}$ but to avoid the proliferation of notation, I'll stick (for now) to $\mathbf{J_z}$.

If you would like to think of all this using matrix notation, then – since the wavefunctions form an infinite-dimensional space – the $\mathbf{J_z}$ matrix would be an infinite-dimensional matrix and the rotations would also be represented by infinite-dimensional matrices.

However, it turns out that these "matrices" can be written in block form. That is, the representation of equation (1) is **reducible**.

What exactly does that mean in practice? In general, the way we express vectors in a vector-space, is to expand them in a basis. E.g.

$$\mathbf{v} = \alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2} + ... + \alpha_N\mathbf{e_N}.$$

When the space is infinite-dimensional, $N = \infty$. The α_i are "the coefficients of the vector $\mathbf{v}$ with respect to the basis $\mathbf{e_i}$. When the vector space is a Hilbert space and thus has an inner product, the basis vectors can be chosen to be orthonormal with respect to the inner product – i.e., in Dirac notation we write $\langle \mathbf{e_i} | \mathbf{e_j} \rangle = \delta_{ij}$.

When a matrix operates on a vector, its effect is to change the coefficients $\alpha_{\mathbf{i}}$ to a new set of coefficients $\beta_{\mathbf{i}}$. For example

$$\mathbf{M} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \tag{2}$$

Now let's say that $\mathbf{M}$ can be written in block form (to be mathematically accurate we should say "can be written in block form relative to the basis $\mathbf{e_i}$".) Then $\mathbf{M}$ would have, for example, the form

$$\mathbf{M} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & b_{11} & b_{12} & b_{13} \\ 0 & b_{21} & b_{22} & b_{23} \\ 0 & b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{3}$$

Suppose we apply this matrix $\mathbf{M}$ to a vector whose last 3 components are 0:

$$\begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The result would be

$$\begin{pmatrix} a_{11}\alpha \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which is again a vector of the same form. In the same way, if $\mathbf{M}$ were applied to a vector whose first component is 0, the result would again be a vector whose first component is 0.

$$\mathbf{M} \begin{pmatrix} 0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ b_{11}\alpha_1 + b_{12}\alpha_2 + b_{13}\alpha_3 \\ b_{21}\alpha_1 + b_{22}\alpha_2 + b_{23}\alpha_3 \\ b_{31}\alpha_1 + b_{32}\alpha_2 + b_{33}\alpha_3 \end{pmatrix} \tag{4}$$

Why is this useful to know? If our matrices form a representation of the rotations, it immediately follows (assuming the same block form for all rotations) that each block also forms a representation of the rotations. Namely, let

$$\mathbf{R}(\boldsymbol{\theta}) = \mathbf{R}(\boldsymbol{\theta_2})\mathbf{R}(\boldsymbol{\theta_1})$$

and let $\mathbf{U}(\boldsymbol{\theta})$ be a representation (i.e.$\mathbf{U}(\boldsymbol{\theta}) = \mathbf{U}(\boldsymbol{\theta_2})\mathbf{U}(\boldsymbol{\theta_1})$) in block form

$$\mathbf{U}(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{U}_{(1)}(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_3(\boldsymbol{\theta}) \end{pmatrix} \tag{5}$$

where $\mathbf{U}_{(1)}(\boldsymbol{\theta})$ is a 1 x 1 matrix and $\mathbf{U}_3(\boldsymbol{\theta})$ is a 3 x 3 matrix. Then, since $\mathbf{U}$ is a representation it's easy to show that

$$\begin{pmatrix} \mathbf{U}_{(1)}(\boldsymbol{\theta}) & \mathbf{0} \\ 0 & \mathbf{U}_3(\boldsymbol{\theta}) \end{pmatrix} = \mathbf{U}(\boldsymbol{\theta_2})\mathbf{U}(\boldsymbol{\theta_1}) = \begin{pmatrix} \mathbf{U}_{(1)}(\boldsymbol{\theta_2})\mathbf{U}_{(1)}(\boldsymbol{\theta_1}) & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_3(\boldsymbol{\theta_2})\mathbf{U}_3(\boldsymbol{\theta_1}) \end{pmatrix} \tag{6}$$

From this we see that $\mathbf{U}_1$ forms a 1 x 1 representation of the rotation group and $\mathbf{U}_3$ forms a 3 x 3 representation of the rotation group. If no similarity transformation can put $\mathbf{U}_3$ into block form, then both $\mathbf{U}_1$ and $\mathbf{U}_3$ are irreducible representations.

To summarize: If a representation can be put into block form, then it is sufficient, in order to 'construct' the entire representation, that we find the individual representations for the blocks. It suffices to focus only on the representations of the infinitesimal generators. It can be demonstrated that the generator $\mathbf{J_z}$ of equation (1), as well as generators $\mathbf{J_y}$ and $\mathbf{J_x}$ can all be written in (the same) block form (relative to a certain basis) where each block is finite.

How do we find a basis that puts the generators in block form? The idea is this: In order that $\mathbf{U}(\boldsymbol{\theta})$ can transform a vector $\begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}^T$ to another vector of the same form, the operation of $\mathbf{U}(\boldsymbol{\theta})$ on the basis vectors $\mathbf{e_2}$, $\mathbf{e_3}$ and $\mathbf{e_4}$ must result in combinations of the same three basis vectors. Recall that each basis vector must be a wave-function and that all three wave-functions must be orthornormal. If we can find three such basis vectors, then we will have succeeded in constructing a 3 x 3 representation of the rotation group. Indeed we can do this, and those 3 vectors turn out to be the spherical harmonics Y_1^{-1}, Y_1^0 and Y_1^1. These are usually expressed in terms of spherical coordinates but, expressing spherical coordinates as functions of cartesian coordinates we get

$$\begin{aligned} Y_1^{-1}(\theta,\phi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x-iy}{r} \\ Y_1^{0}(\theta,\phi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{r} \\ Y_1^{1}(\theta,\phi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{r} \end{aligned} \tag{7}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

These are the well-known $l = 1$ orbitals. Quantum mechanics texts usually go through a complete derivation of representations, but that is for a later time.