# Exercises on Lorentz Symmetry

Bill Celmaster

February 10, 2021

I looked at Zee's problems for section II.3, and believe these may require a lot of work. Here is a multi-part easier exercise.

## 1 Exercise on representations

First recall the following from the earlier notes on rotational symmetries. The 2x2 Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and write $\boldsymbol{\sigma}$ as the 3-tuple $(\sigma_1, \sigma_2, \sigma_3)$.

Then the 2 x 2 matrix representing (up to a phase) a rotation of angle $\theta \equiv |\boldsymbol{\theta}|$ around an axis in the direction of the 3-vector $\boldsymbol{\theta}$ is

$$\mathbf{R}(\boldsymbol{\theta}) = e^{i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta}} = \begin{pmatrix} \cos\frac{\theta}{2} + i\hat{\theta}_3 \sin\frac{\theta}{2} & (i\hat{\theta}_1 + \hat{\theta}_2)\sin\frac{\theta}{2} \\ (i\hat{\theta}_1 - \hat{\theta}_2)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} - i\hat{\theta}_3 \sin\frac{\theta}{2} \end{pmatrix} \tag{2}$$

where the notation $\hat{\theta}_i$ denotes $\frac{\theta_i}{\theta}$.

In this expression, the 2D representation (i.e. $j = \frac{1}{2}$) of the rotation generator $J_i$, is $\frac{\sigma_i}{2}$.

### 1.1 Part A

Recall that we defined $J_{\pm i} = \frac{1}{2}(J_i \pm iK_i)$ and that both $J_{+i}$ and $J_{-i}$ generate the rotation group. Start by considering the Lorentz-group representation denoted by $(0, \frac{1}{2})$. Again, recall that this notation means "$J_-$ is in the 0 (scalar) representation (i.e., $J_{-i}$ are identity matrices)" and "$J_+$ is in the spin-$\frac{1}{2}$ (i.e. 2D) representation".

- Express $J_{+3}$ as a 2 x 2 matrix. (From now on, let's use the terms $J_{\pm i}$ to refer to the **2 x 2 representations of the Lie Algebra** and not to the abstract terms of the Lie Algebra.)

- Let $\nu$ be a 2-vector defined by $\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$. Then compute the new 2-vector $\nu' = J_{+3}\nu$.

  This is not a trick question. It's just a warm-up to get you used to actually doing the matrix operations.

- Next, express $J_{-3}$ as a 2 x 2 matrix operating on 2-vectors $\nu$. Hint: $J_-$ is in the 0 representation, meaning that it doesn't transform vectors.

- We're interested in rotations and boosts, and those are generated by $J_i$ and $K_i$. Find $J_3$ and $K_3$ from our definitions of $J_{\pm 3}$.

- Compute the vector $J_3\nu$.

- Compute $\mathbf{R}(\boldsymbol{\theta})\nu$ using eq.(2), where $\boldsymbol{\theta}$ is the rotation by $\theta$ around the $z$-axis, i.e., $\boldsymbol{\theta} = (0, 0, \theta)$.

- In the last set of notes about Lorentz symmetries, we related rotations to the generators $J_i$ and we **also** related boosts to the generators $K_i$ with the expression, for example, $\mathbf{B}_3(\beta) = e^{\beta K_3}$. Compute $\mathbf{B}_3(\beta)\nu$ for small values of $\beta$ by expanding the Taylor series through the first order in $\beta$.

### 1.2 Part B

This is going to be just like Part A above except that we'll look at the representation denoted by $(\frac{1}{2}, 0)$. Now the roles of $J_-$ and $J_+$ are reversed.

- Using the same methods as in Part A, compute $\mathbf{R}(\boldsymbol{\theta})\nu$ where $\boldsymbol{\theta}$ is the rotation by $\theta$ around the $z$-axis, i.e., $\boldsymbol{\theta} = (0, 0, \theta)$.

- Compute $\mathbf{B}_3(\beta)\nu$ for small values of $\beta$ by expanding the Taylor series through the first order in $\beta$.

- Just for fun. Notice that boosts along the $z$-axis have almost the same form of exponential but without "i". In particular, we have $B_z(\beta) = e^{i\frac{\sigma_3(-i\beta)}{2}}$. This looks just like a rotation except that we've replaced $\theta$ by $-i\beta$. Look at eq. (2) and for simplicity, just concentrate on rotations around the $z$-axis. Then to find boosts along the $z$-axis, replace $\theta$

by $-i\beta$. In case you don't remember it, $\cos(-i\beta) = \cosh(\beta)$. Now compute $\mathbf{B}_3(\beta)\nu$ for general values of $\beta$.

What you should discover, is that rotations transform the same in the two representations $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ but boosts transform differently.

## 2 Constructing a Lorentz-invariant bilinear quantity

The goal here, is to explore ways to construct a Lorentz-invariant quantity out of a bilinear form involving the vector $\nu$ and its conjugate $\nu*$. We'll concentrate exclusively on bilinears of the form $S = \sum_{ij} \nu^{*i} M_{ij} \nu^j$. In the rest of this exercise, I'll leave out the summation sign and instead, use the notation that repeated indices are summed over.

- Let $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $S = \nu^{*i}\nu^i$. Apply to $\nu$ a rotation by $\theta$ around the $z$-axis and show that $S$ does not change.
- Next, apply a boost around the $z$-axis and show that $S$ changes. You can do this either with infinitesimal boosts (so expand to order $\beta$) or with finite boosts. **Notice that your result will depend on whether you use the boost obtained from the $(0, \frac{1}{2})$ or $(\frac{1}{2}, 0)$ representation.**
- The above computations can be shown more abstractly by using dot products and properties of unitarity (and non-unitarity). Since we're doing quantum mechanics, instead of dot-products, use bra-ket notation. In that notation, $S = \langle \nu | \nu \rangle$. The action of rotation around the $z$-axis then becomes $\langle \nu | \mathbf{R}(\theta)^\dagger \mathbf{R}(\theta) | \nu \rangle$. Argue, based on unitarity, that this transformation leaves $S$ invariant. Is a similar thing true for boosts? Explain.

The foregoing example only shows that when $\mathbf{M}$ equals the identity, the resulting bilinear form is not a Lorentz invariant. I can't recall the argument that shows this is true for any choice of self-adjoint $\mathbf{M}$. (Self-adjointness is required for the scalar $\mathbf{S}$ to be real.) But assume that can be shown. That has profound implications since that prohibits the usual kind of mass term that there is in Lagrangians.

One last thing. Suppose we have two vectors $\nu$, which transforms according to representation $(0, \frac{1}{2})$ and $\nu'$ which transforms according to the representation $(\frac{1}{2}, 0)$. Consider $\mathbf{S}' \equiv \nu^{*i}\nu'_i + \nu'^{*i}\nu_i$.

- Show that $\mathbf{S}'$ is a real number (hint: take its complex conjugate.)
- Apply rotations as before and show that **S**' is a Lorentz invariant. The bra-ket notation probably is the easiest way to think about this.
- Now apply boosts. The important thing to observe is that $\nu$ and $\nu'$ transform differently. What is the matrix for boosts in each case? Show that their product is the identity and therefore that $\mathbf{S}'$ is invariant.

What the above shows, is that when there are two 2D fields (i.e. a total of 4 components), one can construct a Lorentz-invariant bilinear and therefore a familiar-looking mass term. Ultimately, since the electron has mass, this is one of the reasons why an electron has 4 components rather than 2.

For many years, it was believed that neutrinos were massless. It was therefore possible for Lorentz-invariant Lagrangians to be constructed from fields with only 2 components. For reasons not yet discussed, those Lagrangians violate the parity symmetry (and parity violations were one of the great discoveries of physics in the past century).