Introduction to Lorentz group representations

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What follows is a set of notes to accompany Zee chapter II.3, the first 2.5 pages (stopping before the section "Spinor representation").

1 Lorentz symmetries

1.1 Rotations around the z-axis with $v = 0$

Rotate a vector by positive θ counter-clockwise around the z-axis. The co-ordinates with superscript 'prime' are the new coordinates of the original vector. The transformation is called *active*.

- $$\begin{aligned} t' &= t \\ x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned} \tag{1}$$

- In matrix notation
$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{2}$$

- The matrix is known as $\mathbf{R}_z(\theta)$.

1.2 Boosts along the z-axis, $v \neq 0$

Take a vector and move it at velocity v relative to the rest frame. Positive v means "going in the +z direction". Then the coordinates ('active') are the coordinates of the new vector.

- Preliminaries
  - Set $c = 1$.
  - $$\gamma_v \equiv \frac{1}{\sqrt{1 - v^2}}$$
- $$\begin{aligned} t' &= \gamma_v(t - vz) \\ x' &= x \\ y' &= y \\ z' &= \gamma_v(z - vt) \end{aligned} \tag{3}$$
- In matrix notation
$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma_v & 0 & 0 & v\gamma_v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma_v & 0 & 0 & \gamma_v \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{4}$$
- The matrix $\mathbf{B}_z$ is called a *boost* in the z direction. It depends on a parameter β defined next.
- **For notational convenience, follow the convention where β is defined as $\beta = \textbf{arcosh}(\gamma_v)$.** Then you can show that the above matrix is
$$\mathbf{B}_z(\beta) = \begin{pmatrix} \cosh\beta & 0 & 0 & \sinh\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix} \tag{5}$$

2 Generators of the Lorentz group

2.1 Generators for z-rotations and z-boosts

Recall that the rotation generators correspond to infinitesimal rotations. Also recall that we care about rotation generators because they provide an efficient way of characterizing and prescribing all rotations. This idea will generalize to the Lorentz group.

- Define $\mathbf{I} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{J}_z \equiv i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then

$$\mathbf{R}_z(\theta) = \mathbf{I} + i\theta\mathbf{J}_z + ...$$

  and more generally

$$\mathbf{R}_z(\theta) = e^{i\theta\mathbf{J}_z}.$$

- Define $\mathbf{K}_z \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Then

$$\mathbf{B}_z(\beta) = \mathbf{I} + \beta\mathbf{K}_z + ... \tag{6}$$

  and more generally

$$\mathbf{B}_z(\beta) = e^{\beta\mathbf{K}_z}.$$

2.2 General rotations and boosts

Everything above can be generalized to the x-axis and the y-axis.

- For notational convenience, it is common to change the indices (t, x, y, z) to indices $(0, 1, 2, 3)$. Then, for example, instead of writing $\mathbf{J}_z$ we write $\mathbf{J}_3$. When we indicate an index for a quadruplet that includes a time-component, we use Greek letters e.g. ν, μ etc. For triplets like (x, y, z) we use Roman letter like i or j.

- We end up with

$$\mathbf{J}_1 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathbf{J}_2 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \mathbf{J}_3 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{K}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{K}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{K}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- Compute $[\mathbf{J}_1, \mathbf{J}_2] = \mathbf{J}_1\mathbf{J}_2 - \mathbf{J}_2\mathbf{J}_1$ to get $-\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = i\mathbf{J}_3$.

- This is a special case of the relationship

$$[\mathbf{J}_i, \mathbf{J}_j] = i\epsilon_{ijk}\mathbf{J}_k$$

which you should recognize as being the Lie Algebra of the rotation group. Since the Lorentz group contains the rotation group, it shouldn't be surprising that the generators of the Lorentz group would include the generators of the rotation group.

- Compute $[\mathbf{K}_1, \mathbf{K}_2] = \mathbf{K}_1\mathbf{K}_2 - \mathbf{K}_2\mathbf{K}_1$ to get $-\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -i\mathbf{J}_3$.

  This is a special case of the relationship

$$[\mathbf{K}_i, \mathbf{K}_j] = -i\epsilon_{ijk}\mathbf{J}_k$$

which looks suggestive but is definitely not the Lie Algebra for the rotation group (in particular, it involves generators for both rotations and boosts).

- In the same way, we get

$$[\mathbf{J}_i, \mathbf{K}_j] = i\epsilon_{ijk}\mathbf{K}_k.$$

2.3 Rewriting the Lie Algebra of the Lorentz group

The preceding commutator relationships define the Lie Algebra of the Lorentz group. But there is a simpler algebra we can write.

- Define two new generators

$$\mathbf{J}_{+i} = \frac{1}{2}(\mathbf{J}_i + i\mathbf{K}_i)$$
$$\mathbf{J}_{-i} = \frac{1}{2}(\mathbf{J}_i - i\mathbf{K}_i)$$

- Notice that

$$\begin{aligned}
[\mathbf{J}_{+i}, \mathbf{J}_{+j}] &= \frac{1}{2}(\mathbf{J}_i + i\mathbf{K}_i)\frac{1}{2}(\mathbf{J}_j + i\mathbf{K}_j) - \frac{1}{2}(\mathbf{J}_j + i\mathbf{K}_j)\frac{1}{2}(\mathbf{J}_i + i\mathbf{K}_i) \\
&= \frac{1}{4}([\mathbf{J}_i\mathbf{J}_j - \mathbf{J}_j\mathbf{J}_i] - [\mathbf{K}_i\mathbf{K}_j - \mathbf{K}_j\mathbf{K}_i] + i[\mathbf{K}_i\mathbf{J}_j - \mathbf{J}_j\mathbf{K}_i] + i[\mathbf{J}_i\mathbf{K}_j - \mathbf{K}_j\mathbf{J}_i]) \\
&= i\frac{1}{4}(\epsilon_{ijk})(\mathbf{J}_k + \mathbf{J}_k + i\mathbf{K}_k + i\mathbf{K}_k) \\
&= i\epsilon_{ijk}\mathbf{J}_{+k}.
\end{aligned}$$

So the new generators $\mathbf{J}_{+i}$ satisfy the same Lie Algebra as the rotation group!

- Similarly $[\mathbf{J}_{-i}, \mathbf{J}_{-j}] = i\epsilon_{ijk}\mathbf{J}_{-k}$, so $\mathbf{J}_{-i}$ **also** satisfies the same Lie Algebra as the rotation group!

- These generators commute, i.e. $[\mathbf{J}_{-i}, \mathbf{J}_{+j}] = 0$, so they are independent.

- We see that there are two independent sets of Lorentz generators which generate the rotation group. Since the Lie Algebra of the rotation group is the same as the Lie Algebra of $SU(2)$, we have finally proven that the algebra of the Lorentz group, sometimes known as $SO(1,3)$ is isomorphic to $su(2) \oplus su(2)$ where lower-case $su(2)$ denotes the Lie Algebra of $SU(2)$.

2.4 General representations of the Lorentz group

With the rotation group, we considered the general question of how to construct linear representations on N-dimensional spaces.

2.4.1 Review of representations of the rotation group

- **Group representations**: The symmetry operations must preserve the essential properties of the symmetries. For example, if you rotate around the y-axis by 90 degrees and then around the x-axis by 90 degrees, that is the same as rotating around the axis $\begin{pmatrix} -1 & -1 & 1 \end{pmatrix}$ by 60 degrees. Thus the operation (representation) for the first rotation followed by the operation for the second, should be the same as the operation for rotation around the axis $\begin{pmatrix} -1 & -1 & 1 \end{pmatrix}$ by 60 degrees.

- **Representation of 360 degree rotation**: In particular, the operation for rotation by 360 degrees **must be the same as doing nothing!** (The identity.)

- Rotations can be represented as N x N matrices operating on N-dimensional vector spaces. **These are called finite-dimensional linear representations of the rotation group, also known as SO(3).**
  - The representations of the rotation group also happen to be unitary.

- The mathematical methodology for classifying representations of the rotation group, is to start with the much easier problem of constructing representations of the **Rotation Lie Algebra. The generators of that Lie Algebra are denoted J_i.**

  - **If there is an NxN representation of rotations**, then if $\tilde{\mathbf{J}}_i$ are the Lie Algebra generators in that representation, then rotations are represented as $e^{i\tilde{\mathbf{J}}\cdot\theta}$.
  - We saw that although the rotation group only has odd-dimension representations(N is odd), there are also even-dimensional representations of the $su(2)$ Lie Algebra which are interesting to us because they constitute *representations up to a phase*, which is all we need in quantum mechanics since states are only defined up to a phase.

- Up to equivalence, there is only one representation for each dimension N. The convention is to label that representation with the integer j such that $N = 2j + 1$. For example, if $j = \frac{1}{2}$, then $N = 2$. j is known as **the spin**.

2.4.2 Symbolic characterization of Lorentz group representations

Remember that the Lie Algebra of the Lorentz group is $su(2) \oplus su(2)$. Therefore the representations are characterized by the representation of each of the $su(2)$ sub-algebras. We label the $(\mathbf{J}_-, \mathbf{J}_+)$ representations as

$$(j_1, j_2).$$

The dimensionality of that representation is $(2j_1+1)(2j_2+1)$. The representations are **not** unitary. (There are no finite-dimensional unitary representations of the Lorentz group.)

For example, if $(j_1, j_2) = (\frac{1}{2}, \frac{1}{2})$, then the total dimensionality is 4. We call this representation *the vector representation.*

As another example, consider $(j_1, j_2) = (\frac{1}{2}, 0)$. This has dimension 2 and is called a *spinor* representation. Notice that $(j_1, j_2) = (0, \frac{1}{2})$ also has dimension 2. It is also called a spinor representation and we distinguish these two inequivalent representations as “left” and “right”. This is different than with the rotation group where there is only one representation (up to equivalence) for each dimension.

2.4.3 Meaning of tensor products

We want to know the meaning of $\mathcal{R}[SU(2)]\otimes\mathcal{R}[SU(2)]$ where $\mathcal{R}[SU(2)]$ means "a representation of the group $SU(2)$".

Let M^1_{ij} be a matrix representation of the first $SU(2)$ and M^2_{ij} be a matrix representation of the second $SU(2)$. Let's say that M^1 acts on an n-dimensional space and is therefore an nxn matrix. Similarly, M^2 acts on an m-dimensional space. Define a vector, V of length nm whose indices are labelled (i, j) and whose values are $V_{ij} = v_{1i} \otimes v_{2j}$. This vector "looks like" a matrix but think of it as one long vector. Then the action of $M^1 \otimes M^2$ on V is $V'_{ij} = \sum_{k,l} M^1_{ik} M^2_{jl} V^{kl}$.