Solutions to exercises on Lorentz Symmetry

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There are many different sign conventions and it's easy to go crazy reconciling them. For now, I will more or less adopt Lancaster's conventions (pp. 337 and 338). That requires changing the problem statement from what I sent out. If you continue to use the original problem statements, I think your answers will be the same as mine if you change θ to $-\theta$ and β to β. [For those paying careful attention, Lancaster's definition of $\mathbf{K}$ differs from mine by a factor of i but the rotations and boosts come out the same in the wash.]

1 Exercise on representations

The 2x2 Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and write $\boldsymbol{\sigma}$ as the 3-tuple $(\sigma_1, \sigma_2, \sigma_3)$.

Then the 2 x 2 matrix representing (up to a phase) a rotation of angle $\theta \equiv |\boldsymbol{\theta}|$ around an axis in the direction of the 3-vector $\boldsymbol{\theta}$ is

$$\mathbf{R}(\boldsymbol{\theta}) = e^{-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta}} = \begin{pmatrix} \cos\frac{\theta}{2} - i\hat{\theta}_3 \sin\frac{\theta}{2} & -(i\hat{\theta}_1 + \hat{\theta}_2)\sin\frac{\theta}{2} \\ -(i\hat{\theta}_1 - \hat{\theta}_2)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + i\hat{\theta}_3 \sin\frac{\theta}{2} \end{pmatrix} \tag{2}$$

where the notation $\hat{\theta}_i$ denotes $\frac{\theta_i}{\theta}$.

In this expression, the 2D representation (i.e. $j = \frac{1}{2}$) of the rotation generator J_i, is $\frac{\sigma_i}{2}$.

1.1 Part A

Recall that we defined $J_{\pm i} = \frac{1}{2}(J_i \pm iK_i)$ and that both J_{+i} and J_{-i} generate the rotation group. Start by considering the Lorentz-group representation denoted by $(0, \frac{1}{2})$. Again, recall that this notation means "J_- is in the 0 (scalar) representation (i.e., J_{-i} are 0)" and "J_+ is in the spin-$\frac{1}{2}$ (i.e. 2D) representation".

- Express J_{+3} as a 2 x 2 matrix. (From now on, let's use the terms $J_{\pm i}$ to refer to the **2 x 2 representations of the Lie Algebra** and not to the abstract terms of the Lie Algebra.)

  SOLUTION: $J_{+3} = \frac{\sigma_3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$.

- Let ν be a 2-vector defined by $\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$. Then compute the new 2-vector $\nu' = J_{+3}\nu$.

  This is not a trick question. It's just a warm-up to get you used to actually doing the matrix operations.

  SOLUTION: $\nu' = J_{+3}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \frac{\nu_1}{2} \\ -\frac{\nu_2}{2} \end{pmatrix}$.

- Next, express J_{-3} as a 2 x 2 matrix operating on 2-vectors ν. Hint: J_- is in the 0 representation, meaning that it doesn't transform vectors.

  SOLUTION: $J_{-3} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Apologies for the hint. It's more confusing than helpful. In the 0-representation, rotations don't transform vectors so they are represented by the identity. An infinitesimal rotation is therefore the identity and the generator must be 0.

- We're interested in rotations and boosts, and those are generated by J_i and K_i. Find J_3 and K_3 from our definitions of $J_{\pm 3}$.

  SOLUTION: Recall that $J_{\pm i} = \frac{1}{2}(J_i \pm iK_i)$. Then $J_3 = J_{+3} + J_{-3} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$. $K_3 = -i\,(J_{+3} - J_{-3}) = -i\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$.

- Compute the vector $J_3\nu$.

  SOLUTION: Since J_3 looks just like J_{+3} we can copy the previous result. Namely $J_3\nu = \begin{pmatrix} \frac{\nu_1}{2} \\ -\frac{\nu_2}{2} \end{pmatrix}$.

- Compute $\mathbf{R}(\boldsymbol{\theta})\nu$ using eq.(2), where $\boldsymbol{\theta}$ is the rotation by θ around the z-axis, i.e., $\boldsymbol{\theta} = (0, 0, \theta)$.

  SOLUTION: For this rotation around the z-axis, we have $\mathbf{R}(\theta) = e^{-iJ_3\theta} = e^{-i\frac{\sigma_3}{2}\theta} = \begin{pmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \end{pmatrix}$. Then $\mathbf{R}(\theta)\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\theta}{2}}\nu_1 \\ e^{i\frac{\theta}{2}}\nu_2 \end{pmatrix}$.

- In the last set of notes about Lorentz symmetries, we related rotations to the generators J_i and we **also** related boosts to the generators K_i with the expression, for example, $\mathbf{B}_3(\beta) = e^{i\beta K_3}$. Compute $\mathbf{B}_3(\beta)\nu$ for small values of β by expanding the Taylor series through the first order in β.

  SOLUTION:[1] $\mathbf{B}_3(\beta) = \mathbf{I} + i\beta\mathbf{K}_3 + ... = \mathbf{I} + \beta\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + ...$

1.2 Part B

This is going to be just like Part A above except that we'll look at the representation denoted by $(\frac{1}{2}, 0)$. Now the roles of J_- and J_+ are reversed.

- Using the same methods as in Part A, compute $\mathbf{R}(\boldsymbol{\theta})\nu$ where $\boldsymbol{\theta}$ is the rotation by θ around the z-axis, i.e., $\boldsymbol{\theta} = (0, 0, \theta)$.

  SOLUTION: Now $J_{-3} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ and $J_{+3} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ so $J_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$. This is the same as we had before, so again, $\mathbf{R}(\theta)\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\theta}{2}}\nu_1 \\ e^{i\frac{\theta}{2}}\nu_2 \end{pmatrix}$.

- Compute $\mathbf{B}_3(\beta)\nu$ for small values of β by expanding the Taylor series through the first order in β.

  SOLUTION: First we need to compute $\mathbf{K}_3$. Recall from above that $\mathbf{K}_3 = -i(\mathbf{J}_{+3} - \mathbf{J}_{-3})$. Using the $\mathbf{J}_{\pm 3}$ matrices for this representation,

[1] I've discovered that somewhere along the line I've made a sign error in the definition of **K**. I'm too lazy right now to track it down, but beware. The key points are all OK, but the details of signs need to be sorted out!!

we have $K_3 = -i\left(\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\right) = i\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}$. We have the expansion derived above for $\mathbf{B}_3(\beta)$ thus leading to $\mathbf{B}_3(\beta) = \mathbf{I} - \beta\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix} + ...$

- Just for fun. Notice that boosts along the z-axis have almost the same form of exponential but without "i". In particular, we have $B_z(\beta) = e^{i\frac{\sigma_3(-i\beta)}{2}}$. This looks just like a rotation except that we've replaced θ by $-i\beta$. Look at eq. (2) and for simplicity, just concentrate on rotations around the z-axis. Then to find boosts along the z-axis, replace θ by $-i\beta$. In case you don't remember it, $\cos(-i\beta) = \cosh(\beta)$. Now compute $\mathbf{B}_3(\beta)\nu$ for general values of β.

  SOLUTION: We derived before, for the $(0, \frac{1}{2})$ representation, that the rotation $\mathbf{R}(\theta) = e^{i\mathbf{J}_3\theta} = e^{i\frac{\sigma_3}{2}\theta} = \begin{pmatrix}\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} & 0\\ 0 & \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix}e^{i\frac{\theta}{2}} & 0\\ 0 & e^{-i\frac{\theta}{2}}\end{pmatrix}$. In the $(\frac{1}{2}, 0)$ representation, $\mathbf{J}_3$ is again equal $i\frac{\sigma_3}{2}$ so the rotation looks exactly the same.

  In the $(0, \frac{1}{2})$ representation, the boost $\mathbf{B}_3(\beta) = e^{iK_3\beta} = e^{\frac{\sigma_3}{2}\beta}$. This is the same as replacing, in the rotation expression, θ by $-i\beta$. So $\mathbf{B}_3(\beta) = \begin{pmatrix}e^{i\frac{-i\beta}{2}} & 0\\ 0 & e^{-i\frac{-i\beta}{2}}\end{pmatrix} = \begin{pmatrix}e^{\frac{\beta}{2}} & 0\\ 0 & e^{-\frac{\beta}{2}}\end{pmatrix}$. On the other hand, in the $(\frac{1}{2}, 0)$ representation, we found that $\mathbf{K}_3 = i\frac{\sigma_3}{2}$, so $\mathbf{B}_3(\beta) = e^{-\sigma_3\beta}$. From this, we get $\mathbf{B}_3(\beta) = \begin{pmatrix}e^{-\frac{\beta}{2}} & 0\\ 0 & e^{\frac{\beta}{2}}\end{pmatrix}$. Just for grins, we can compute, in that reference frame, $\mathbf{B}_3(\beta)\nu = \begin{pmatrix}e^{-\frac{\beta}{2}}\nu_1\\ e^{\frac{\beta}{2}}\nu_2\end{pmatrix}$.

  Finally, let's tie this to hyperbolic functions. Recall the definitions of cosh and sinh. $\cosh(\alpha) = \frac{e^{\alpha}+e^{-\alpha}}{2}$ and $\sinh(\alpha) = \frac{e^{\alpha}-e^{-\alpha}}{2}$. Then using these functions, we find $\mathbf{B}_3(\beta) = \frac{1}{2}\begin{pmatrix}\cosh(-\frac{\beta}{2}) + \sinh(-\frac{\beta}{2}) & 0\\ 0 & \cosh(-\frac{\beta}{2}) - \sinh(-\frac{\beta}{2})\end{pmatrix}$.

  To summarize, rotations transform the same in the two representations, but boosts transform differently ($\beta \to -\beta$.). Furthermore, boosts can be thought of a analytic continuations of rotations, where the angle is rotated to an imaginary angle.

2 Constructing a Lorentz-invariant bilinear quantity

The goal here, is to explore ways to construct a Lorentz-invariant quantity out of a bilinear form involving the vector ν and its conjugate ν^*. We'll concentrate exclusively on bilinears of the form $S = \sum_{ij} \nu_i^* M^{ij} \nu_j$.[2] In the rest of this exercise, I'll leave out the summation sign and instead, use the notation that repeated indices are summed over.

- Let $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $S = \nu_i^* \nu_i$. Apply to ν a rotation by θ around the z-axis and show that S does not change.

  SOLUTION: $\nu^* = \begin{pmatrix} \nu_1^* \\ \nu_2^* \end{pmatrix}$. Define $\nu' = \mathbf{R}(\theta)\nu$ so $\nu'^* = (\mathbf{R}(\theta)\nu)^*$. Above, we derived $\nu' = \mathbf{R}(\theta)\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\theta}{2}}\nu_1 \\ e^{i\frac{\theta}{2}}\nu_2 \end{pmatrix}$ so $\nu'^* = \begin{pmatrix} e^{i\frac{\theta}{2}}\nu_1^* \\ e^{-i\frac{\theta}{2}}\nu_2^* \end{pmatrix}$. We see that $\nu_i'^* \nu_i' = \nu_i^* \nu_i$. Therefore $S' = S$, which was to be shown.

- Next, apply a boost around the z-axis and show that S changes. You can do this either with infinitesimal boosts (so expand to order β) or with finite boosts. **Notice that your result will depend on whether you use the boost obtained from the $(0, \frac{1}{2})$ or $(\frac{1}{2}, 0)$ representation.**

  SOLUTION: Above, we derived $\nu' = \mathbf{B}_3(\beta)\nu = \begin{pmatrix} e^{-\frac{\beta}{2}}\nu_1 \\ e^{\frac{\beta}{2}}\nu_2 \end{pmatrix}$. Then $\nu'^* = \begin{pmatrix} e^{-\frac{\beta}{2}}\nu_1^* \\ e^{\frac{\beta}{2}}\nu_2^* \end{pmatrix}$. So $\nu_1'^* \nu_1' = e^{-\beta}\nu_1^*\nu_1$ and $\nu_2'^* \nu_2' = e^{\beta}\nu_2^*\nu_2$ from which we see that $S' = e^{-\beta}\nu_1^*\nu_1 + e^{\beta}\nu_2^*\nu_2$, compared to $S = \nu_2^*\nu_2 + \nu_1^*\nu_1$.
  In the other representation, we would get $S' = e^{\beta}\nu_1^*\nu_1 + e^{-\beta}\nu_2^*\nu_2$.

- The above computations can be shown more abstractly by using dot products and properties of unitarity (and non-unitarity). Since we're doing quantum mechanics, instead of dot-products, use bra-ket notation. In that notation, $S = \langle \nu | \nu \rangle$. The action of rotation around the z-axis then becomes $\langle \nu | \mathbf{R}(\theta)^\dagger \mathbf{R}(\theta) | \nu \rangle$. Argue, based on unitarity,

[2] In the original statement of the exercises, I reversed the upper and lower indices. Ultimately that doesn't matter but I found it easier to read when I reversed them.

that this transformation leaves S invariant. Is a similar thing true for boosts? Explain.

SOLUTION: From earlier, $\mathbf{R}(\theta) = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$. A unitary matrix $\mathbf{U}$ has the property that $\mathbf{U}^{-1} = \mathbf{U}^{\dagger}$. $\mathbf{R}^{\dagger}(\theta) = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$. If you multiply this by $\mathbf{R}(\theta)$ you'll get the identity. So that shows $\mathbf{R}^{\dagger}(\theta) = \mathbf{R}^{-1}(\theta)$ and therefore that rotations are unitary. Finally, $\langle\nu|\mathbf{R}(\theta)^{\dagger}\mathbf{R}(\theta)|\nu\rangle = \langle\nu|\nu\rangle$.

This doesn't work for boosts. Recall $\mathbf{B}_3(\beta) = \begin{pmatrix} e^{-\frac{\beta}{2}} & 0 \\ 0 & e^{\frac{\beta}{2}} \end{pmatrix}$. It's easy to see that $\mathbf{B}_3^{-1}(\beta) = \begin{pmatrix} e^{\frac{\beta}{2}} & 0 \\ 0 & e^{-\frac{\beta}{2}} \end{pmatrix}$ **Notice for future reference that $\mathbf{B}_3^{-1}$ is the same matrix as what we'd get for the boost in the OTHER representation!)** but $\mathbf{B}_3^{\dagger}(\beta) = \begin{pmatrix} e^{-\frac{\beta}{2}} & 0 \\ 0 & e^{\frac{\beta}{2}} \end{pmatrix}$ so $\mathbf{B}_3^{-1} \neq \mathbf{B}_3^{\dagger}$ therefore boosts are **not** unitary. We shouldn't be too surprised, since it's known that there are no finite-dimensional unitary representations of the Lorentz group. Then $\langle\nu|\mathbf{B}_3(\beta)^{\dagger}\mathbf{B}_3(\beta)|\nu\rangle = \langle\nu|\mathbf{B}_3(2\beta)\nu\rangle \neq \langle\nu|\nu\rangle$.

The foregoing example only shows that when $\mathbf{M}$ equals the identity, the resulting bilinear form is not a Lorentz invariant. I can't recall the argument that shows this is true for any choice of self-adjoint $\mathbf{M}$. (Self-adjointness is required for the scalar $\mathbf{S}$ to be real.) But assume that can be shown. That has profound implications since that prohibits the usual kind of mass term that there is in Lagrangians.

One last thing. Suppose we have two vectors ν, which transforms according to representation $(0, \frac{1}{2})$ and ν' which transforms according to the representation $(\frac{1}{2}, 0)$. Consider $\mathbf{S}' \equiv \nu^{*i}\nu'_i + \nu'^{*i}\nu_i$.

- Show that $\mathbf{S}'$ is a real number (hint: take its complex conjugate.)

  SOLUTION: $\mathbf{S}'^{*} = \left(\nu^{*i}\nu'_i + \nu'^{*i}\nu_i\right)^* = \left(\nu^{i}\nu'^{*}_i + \nu'^{i}\nu^{*}_i\right) = \left(\nu_i\nu'^{*i} + \nu'_i\nu^{i*}\right) = \mathbf{S}$.

- Apply rotations as before and show that $\mathbf{S}$' is a Lorentz invariant. The bra-ket notation probably is the easiest way to think about this.

SOLUTION: $\mathbf{S} = \langle\nu|\nu'\rangle + \langle\nu'|\nu\rangle$. Then $\mathbf{S}' = \langle\nu|\mathbf{R}(\theta)^\dagger\mathbf{R}(\theta)|\nu'\rangle + \langle\nu'|\mathbf{R}(\theta)^\dagger\mathbf{R}(\theta)|\nu\rangle = \langle\nu|\nu'\rangle + \langle\nu'|\nu\rangle = \mathbf{S}$. We have employed the fact that rotations are the same in both representations.

- Now apply boosts. The important thing to observe is that ν and ν' transform differently. What is the matrix for boosts in each case? Show that their product is the identity and therefore that $\mathbf{S}'$ is invariant.

  SOLUTION: We mentioned before that a boost in the $(0, \frac{1}{2})$ representation transforms $|\nu\rangle$ to $e^{\frac{\sigma_2}{2}}|\nu\rangle$ and that a boost in the $(\frac{1}{2}, 0)$ representation transforms $|\nu'\rangle$ to $e^{-\frac{\sigma_2}{2}}|\nu\rangle$. In both cases, the boosts are self-adjoint. Therefore S transforms to $\langle\nu|\mathbf{B}_3(\beta)^\dagger\mathbf{B}_3'(\beta)|\nu'\rangle + \langle\nu'|\mathbf{B}_3'(\beta)^\dagger\mathbf{B}_3(\beta)|\nu\rangle = \langle\nu|e^{\frac{\sigma_2}{2}}e^{-\frac{\sigma_2}{2}}|\nu'\rangle + \langle\nu'|e^{-\frac{\sigma_2}{2}}e^{\frac{\sigma_2}{2}}|\nu\rangle = \langle\nu|\nu'\rangle + \langle\nu'|\nu\rangle = \mathbf{S}$ where the *prime* over $\mathbf{B}_3$ denotes that the boost is in the representation $(\frac{1}{2}, 0)$.

What the above shows, is that when there are two 2D fields (i.e. a total of 4 components), one can construct a Lorentz-invariant bilinear and therefore a familiar-looking mass term. Ultimately, since the electron has mass, this is one of the reasons why an electron has 4 components rather than 2.

For many years, it was believed that neutrinos were massless. It was therefore possible for Lorentz-invariant Lagrangians to be constructed from fields with only 2 components. For reasons not yet discussed, those Lagrangians violate the parity symmetry (and parity violations were one of the great discoveries of physics in the past century).