Exercise on scalar particle creation

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The exercise will be for you to derive an expression for a normalized scalar state with energy 3ω. This state will represent a state with 3 particles each of energy ω. Up to now, we've been dealing with fermions and anticommutation relations, and for those particles, it's not possible to have more than one state with energy ω.

Notation and assumptions:

1. The normalized vacuum state is $|\Omega\rangle$.
2. The annihilation operator a annihilates the vacuum. That is, $a|\Omega\rangle = 0$.
3. The scalars have a commutation relation $aa^\dagger - a^\dagger a = [a, a^\dagger] = 1$.
4. The energy operator $\hat{E} = \omega a^\dagger a$.

I will derive expressions for the normalized 1-particle and 2-particle states and then you can use the same procedure to derive the 3-particle state.

- Assertion: The state $|1\rangle = a^\dagger|\Omega\rangle$ has an energy of ω.

  Proof: Apply the energy operator to that state and show its eigenvalue is ω.

  $$\begin{aligned} \hat{E}a^\dagger|\Omega\rangle = \left(\omega a^\dagger a\right) a^\dagger|\Omega\rangle &= \omega a^\dagger a a^\dagger|\Omega\rangle \\ &= \omega a^\dagger \left(a^\dagger a|\Omega\rangle + |\Omega\rangle\right) \\ &= \omega a^\dagger|\Omega\rangle, \end{aligned} \tag{1}$$

  where the second line follows by applying assumption #3 and the last line follows by applying assumption #2.

- Assertion: The state $|1\rangle = a^\dagger|\Omega\rangle$ is normalized.

Proof: We want to show that $\langle 1|1\rangle = 1$. Observe that $\langle 1| = \langle\Omega|a$. Thus $\langle 1|1\rangle = \langle\Omega|aa^\dagger|\Omega\rangle$. But then

$$\begin{aligned} \langle\Omega|aa^\dagger|\Omega\rangle &= \langle\Omega|a^\dagger a + 1|\Omega\rangle \\ &= \langle\Omega|\Omega\rangle \\ &= 1, \end{aligned} \tag{2}$$

where the first line follows from assumption #3, the second line follows from assumption #2 and the last line follows from assumption #1.

- Assertion: The state $|2\rangle = \frac{1}{\sqrt{2}}a^\dagger|1\rangle$ has an energy of 2ω.

  Proof: Expand $a^\dagger|1\rangle$ to $a^\dagger a^\dagger|\Omega\rangle$ and then apply the energy operator to that state and show its eigenvalue is 2ω.

$$\begin{aligned} \hat{E}a^\dagger a^\dagger|\Omega\rangle = \left(\omega a^\dagger a\right) a^\dagger a^\dagger|\Omega\rangle &= \omega a^\dagger a a^\dagger a^\dagger|\Omega\rangle \\ &= \omega a^\dagger\left(a^\dagger a a^\dagger|\Omega\rangle + a^\dagger|\Omega\rangle\right) \\ &= \omega a^\dagger a^\dagger\left(a^\dagger a|\Omega\rangle + |\Omega\rangle\right) + \omega a^\dagger a^\dagger|\Omega\rangle \\ &= \omega a^\dagger a^\dagger|\Omega\rangle + \omega a^\dagger a^\dagger|\Omega\rangle \\ &= 2\omega a^\dagger a^\dagger|\Omega\rangle, \end{aligned} \tag{3}$$

  where the second line follows from assumption #3 and so does the third line, and the fourth line follows from assumption #2. Then, since $a^\dagger a^\dagger|\Omega\rangle$ is an eigenvector of $\hat{E}$ with eigenvalue 2ω, the same is true if you multiply that state by a constant, and thus the state $|2\rangle$ has energy 2ω.

- Assertion: The state $|2\rangle = \frac{1}{\sqrt{2}}a^\dagger a^\dagger|\Omega\rangle$ is normalized.

  Proof: We want to show that $\langle 2|2\rangle = 1$. Observe that $\langle 2| = \frac{1}{\sqrt{2}}\langle\Omega|aa$. Thus $\langle 2|2\rangle = \frac{1}{2}\langle\Omega|aaa^\dagger a^\dagger|\Omega\rangle$. But then

$$\begin{aligned} \frac{1}{2}\langle\Omega|aaa^\dagger a^\dagger|\Omega\rangle &= \frac{1}{2}\langle\Omega|aa^\dagger aa^\dagger + aa^\dagger|\Omega\rangle \\ &= \frac{1}{2}\langle\Omega|aa^\dagger a^\dagger a + aa^\dagger + aa^\dagger|\Omega\rangle \\ &= \frac{1}{2}\langle\Omega|2aa^\dagger|\Omega\rangle \\ &= \langle\Omega|a^\dagger a + 1|\Omega\rangle \\ &= \langle\Omega|\Omega\rangle \\ &= 1, \end{aligned} \tag{4}$$

where the first and second lines employ assumption #3, the third line employs assumption #2, the fourth line employs assumption #3, the fifth line employs assumption #2 and the last line employs assumption #1.

EXERCISE: Show that the state $|3\rangle = \frac{1}{\sqrt{3}} a^\dagger |2\rangle$ is normalized and has an energy of 3ω.