# Solving the Dirac equation for neutrinos

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## 1 Preface

There seem to be a countless number of articles and class-notes about the theory of fermions and, in particular, about solutions and behaviors of various forms of the Dirac equation. Many of the articles are lengthy and detailed. The authors often appear motivated by a conviction that other articles have been either unclear, too complicated, ambiguous or simply incomplete. Part of the reason for all this, is that the theory is rife with conventions and notations. Conclusions derived in one convention cannot necessarily be expressed in the same way, using a different convention. This problem appears especially confusing with regards to the term "Majorana". There are 3 distinct, but entangled, uses of that term:

- Majorana particles (the particle is self-charge-conjugate)
- Majorana mass (a mass term for a 2-spinor theory)
- Majorana representation (a form of the Dirac matrices)

The notes which follow are my attempt to address a few questions for which published answers – appearing in multiple papers – were unsatisfactory to me. In retrospect, the unsatisfactory nature of these answers was, in large part, because they required far more detail than I had imagined necessary. In the end, I've been unable to write anything more streamlined. So, although I'm now satisfied that I understand the answers to my original question, I'm convinced that my notes will ultimately be as obtuse to others, as other notes have been to me. Certainly, it's highly unlikely that anything novel is contained in my notes. Somewhere, there is undoubtedly a paper treating the subject in almost identical fashion to what I've done. But maybe what was necessary for me, is to have derived everything myself from scratch.

# 2 Introduction

These notes address some terminology concerning the masses of neutrinos. A question is frequently asked, "are neutrinos Majorana particles, or are they Dirac particles?" I would propose that an alternate way of asking the question is "are neutrinos Dirac particles *as well as* Majorana particles?" The second formulation of the question suggests that there's nothing interesting about the Majorana nature of neutrinos. What would be interesting is if they are also Dirac.

The point is this. In order for neutrinos to obtain mass, there needs to be at least two distinct particles. I will explain that point shortly. It's possible to write a theory where each particle is its own antiparticle. However, if those two distinct particles have the same mass, then the theory can be written so that one particle is regarded as the (distinct) antiparticle of the other. This would be the usual 4-spinor Dirac formulation. Thus the equal-mass situation is just a special case of the two-Majorana theory.

These notes start with a thorough review of the theory of a single Majorana field. We will see that this theory is equivalent to the theory of a single two-spinor field with a Majorana mass term. The notes then continue by examining a theory with both a left and right chiral field, and with a Dirac mass term (mixing the fields) as well as a Majorana mass term for one of the fields. This theory is currently regarded as the most general neutrino theory consistent with the discovery of neutrino masses. We'll see that this theory can be written as a theory of two Majorana fields.

A natural question might be "why does the electron field happen to be the special case of a Dirac field?" An answer – probably the standard answer – is that electrons carry charge, so gauge invariance forces the Dirac "condition". Neutrinos might not be similarly constrained, hence allowing the more general physics of two Majorana particles of separate masses.

# 3 Plan

- The canonical Dirac equation and its solutions
- Equations of motion for a single-Majorana theory
- Obtaining equations of motion from canonical Lagrangians (used to normalize the annihilation and creation operators)
- Obtain and solve the equations of motion for the weak interaction neutrino kinetic terms, in terms of Majorana fields

- Rewrite the original fields which appear in the weak-interaction Lagrangian, in terms of the Majorana fields
- Derive the conclusions in a state formulation by converting from the field formulation.

A notational comment: In pure math, vectors are essentially just objects with components, and those objects can be added and maybe can even participate in dot products. In that sense, a spinor is an example of a vector. In physics, we tend to use the word "vector" to also implicitly include particular transformation properties (usually fundamental representations of rotations or Lorentz transformations). The word "spinor" is used for "mathematical vectors" that transform according to the spinor representation. One of the quantities I introduce below, has 8 components. I simply call it an "object" but again, it's just a type of vector. No guarantees that I've managed below to keep track of my usage of "vectors" and "spinors".

# 4 The canonical Dirac equation and its solutions

Since a Majorana spinor is defined as a constrained form of a Dirac spinor solution, we'll start by describing those solutions, $\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$, of the "canonical" Dirac equation. By "canonical", I mean a specific (and familiar) form of the 4-spinor Dirac equation where, amongst other conventions, the mass parameter $m$ is positive.

$$i\partial\!\!\!/\psi - m\psi = 0. \tag{1}$$

The $\gamma$ matrices need only satisfy the Dirac algebra, but for definiteness, we'll work in the Weyl (chiral) basis where the $\gamma$ matrices are defined in terms of the Pauli matrices $\sigma_i$ $\left(\text{for example,}\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{2}$$

The importance of this setup, is that it permits borrowing from other texts such as Schwartz, the entire formalism of the Dirac theory including classical solutions and, importantly, the annihilation-creation operators and their particle interpretations.

The Dirac equation is solved by starting with the ansatz

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}^m}} \left( a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx} \right) , \tag{3}$$

where $\omega_{\mathbf{p}}^m = \sqrt{\mathbf{p}^2 + m^2} = p_0$. In the above expression, for each index value $s$, $u^s(p)$ and $v^s(p)$ have 4 complex components. $a_p^s$ and $b_p^s$ each have 1 complex component and will eventually be interpreted as operators. We'll begin by solving the equation classically and will treat $a_p^s$ and $b_p^s$ as complex coefficients (and these, in turn, depend on the choices we make for $u^s(p)$ and $v^s(p)$). Eventually, as we promote the Dirac spinors to operator-fields for the quantum theory, then $\psi$, $a_p^s$ and $b_p^s$ will all be (non-commutative) operators on the Hilbert space. The quantization condition will impose anti-commutation rules on the fields $\psi$ and $\psi^\dagger$. These, in turn will impose anti-commutation relations on $a_p^s$, $b_p^s$ and their adjoints. With the right choices of $u^s(p)$ and $v^s(p)$, we will then be able to interpret $a_p^s$ and $b_p^s$ as the annihilation operators of particles and antiparticles respectively. We will refer to those "right" choices as "canonical solutions."

Canonical solutions (with the above ansatz) are given, for example, in Schwartz[1] p. 190. It is usual to only show solutions for motion in the $z$-direction, However, those solutions are real, therefore making it difficult to extrapolate general behaviors under conjugation (of course, one could choose to maintain throughout, the most general solutions for any momentum, but I prefer something more concrete). So, instead of the $z$-direction, obtain solutions for 4-momenta $p^\mu = (E, 0, p_y, 0)$. The derivation and generalizations are given in Appendix A.

$$u^1(m,p) = \frac{1}{2} \begin{pmatrix} \sqrt{\omega_{\mathbf{p}}^m + p_y} + \sqrt{\omega_{\mathbf{p}}^m - p_y} \\ i\left(\sqrt{\omega_{\mathbf{p}}^m - p_y} - \sqrt{\omega_{\mathbf{p}}^m + p_y}\right) \\ \sqrt{\omega_{\mathbf{p}}^m + p_y} + \sqrt{\omega_{\mathbf{p}}^m - p_y} \\ i\left(\sqrt{\omega_{\mathbf{p}}^m + p_y} - \sqrt{\omega_{\mathbf{p}}^m - p_y}\right) \end{pmatrix}, u^2(m,p) = \frac{1}{2} \begin{pmatrix} i\left(\sqrt{\omega_{\mathbf{p}}^m + p_y} - \sqrt{\omega_{\mathbf{p}}^m - p_y}\right) \\ \sqrt{\omega_{\mathbf{p}}^m + p_y} + \sqrt{\omega_{\mathbf{p}}^m - p_y} \\ i\left(\sqrt{\omega_{\mathbf{p}}^m - p_y} - \sqrt{\omega_{\mathbf{p}}^m + p_y}\right) \\ \sqrt{\omega_{\mathbf{p}}^m + p_y} + \sqrt{\omega_{\mathbf{p}}^m - p_y} \end{pmatrix}$$

$$v^1(m,p) = \frac{1}{2} \begin{pmatrix} \sqrt{\omega_{\mathbf{p}}^m + p_y} + \sqrt{\omega_{\mathbf{p}}^m - p_y} \\ i\left(\sqrt{\omega_{\mathbf{p}}^m - p_y} - \sqrt{\omega_{\mathbf{p}}^m + p_y}\right) \\ -\sqrt{\omega_{\mathbf{p}}^m + p_y} - \sqrt{\omega_{\mathbf{p}}^m - p_y} \\ -i\left(\sqrt{\omega_{\mathbf{p}}^m + p_y} - \sqrt{\omega_{\mathbf{p}}^m - p_y}\right) \end{pmatrix}, v^2(m,p) = \frac{1}{2} \begin{pmatrix} i\left(\sqrt{\omega_{\mathbf{p}}^m + p_y} - \sqrt{\omega_{\mathbf{p}}^m - p_y}\right) \\ \sqrt{\omega_{\mathbf{p}}^m + p_y} + \sqrt{\omega_{\mathbf{p}}^m - p_y} \\ -i\left(\sqrt{\omega_{\mathbf{p}}^m - p_y} - \sqrt{\omega_{\mathbf{p}}^m + p_y}\right) \\ -\sqrt{\omega_{\mathbf{p}}^m + p_y} - \sqrt{\omega_{\mathbf{p}}^m - p_y} \end{pmatrix}$$

For notational convenience, we will drop the argument $m$, so $u^1(p) \equiv u^1(m,p)$ etc.

[1] I have the second printing but there was an erratum posted, which has the basis vectors shown here.

We will also write out the Dirac equation and its components in terms of the upper two and lower two components of spinors. To do this, introduce some extra notation. Distinguish the "up" and "down" halves of a generic spinor $\psi$ with the notation $\psi^U = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ and $\psi^D = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$. Also, it will be convenient to define $\sigma^\mu$ as $(\mathbf{I}_{2\text{x}2}, \sigma_1, \sigma_2, \sigma_3)$ and $\bar{\sigma}^\mu$ as $(\mathbf{I}_{2\text{x}2}, -\sigma_1, -\sigma_2, -\sigma_3)$. Then the Dirac equation eq. (1) becomes

$$\begin{aligned} i\partial_\mu \sigma^\mu \psi^D - m\psi^U = 0, \\ i\partial_\mu \bar{\sigma}^\mu \psi^U - m\psi^D = 0. \end{aligned} \tag{5}$$

# 5 Fields, particles, states and wavefunctions

Different treatments of the subject matter, focus on different objects that are related to one another. We've touched on these, but in this section I'll try to distinguish and compare those objects.

## 5.1 Fields

A field-theoretic description starts with a Lagrangian (whose integral is called the action) written in terms of fields, that are generically written $\psi^i(x)$ where $i$ is an index and $x$ is the time-space coordinate vector.

Although we often say that $\psi^i(x)$ is an operator, I think that's jumping ahead of ourselves. In my opinion, a better way of proceeding is to say that the fields in the Lagrangian are complex-valued functions,[2] and then proceed to find the extrema of the action by writing a Euler-Lagrange equation **for complex-valued functions**.

However, AFTER writing the E-L equation but before solving it, we *promote* the fields to operators. Formally they continue to be written as $\psi^i(x)$ but now those quantities are understood as operator-valued functions and the free variables of the E-L solutions become operators.

## 5.2 Particles and single-particle states

We need one more ingredient before we can make sense of the operators. After all, an operator has to act on something, and we've said nothing about what is acted upon. Miraculously, it turns out that all we need to say about the field operators is how they (anti)commute with one another! From there,

[2] strictly speaking, for fermions, we need to extend all this to the rather abstract concept of Grassman-variable-valued functions but for most of this discussion we can ignore that

we can construct a vector space (more specifically, a Hilbert space) and a set of operators on that space, so that those operators obey the E-L equations AND the (anti)commutation relations. This construction is a concrete realization of the abstract field theory and is referred-to as a *representation*. The representation can (usually?) be made unique by asserting there is only a single lowest-energy state (the energy is an eigenvalue of the Hamiltonian operator, which can be constructed out of the operator fields) – an assertion that cannot be obeyed in theories with broken symmetries.

Rather than showing the path that takes us all the way from the Lagrangian to the above-mentioned representation of the operators, I'll zero in on the result. It turns out that the solution of the E-L equation, in operator form, can be expanded in terms of a few operators traditionally described as $a_s(p)$ and $b_s(p)$. These have well-known (anti)commutation relations amongst one another and are known respectively as particle-annihilation operators and antiparticle-annihilation operators. Their adjoints are known as particle and antiparticle creation operators.

***Now I'll define what I'm going to mean by the words 'particle' and 'antiparticle' and their single-particle states***. First, let's give a name to the kind of particle associated with the field $\psi$. I'll call it $\nu$. It's just a name but I want to keep it distinct from the name of the field! We apply the particle creation operator to the vacuum (state) and define the resultant state as $|\nu, s, p\rangle = a_s^\dagger(p)|0\rangle$. ***I will refer to*** $|\nu, s, p\rangle$ ***as a*** $\nu$ ***single-particle-state and I will refer to the multiplicity of those states (the collection for all spin*** $s$ ***and momenta*** $p$***) as the*** $\nu$***-particle!*** Similar remarks pertain to antiparticles. By convention, if $\nu$ is the name of the particle associated with the field $\psi$, then $\bar{\nu}$ is the name of the antiparticle associated with with the field $\psi$ and we write $|\bar{\nu}, s, p\rangle = b_s^\dagger(p)|0\rangle$. I will refer to $|\bar{\nu}, s, p\rangle$ as a $\bar{\nu}$ single-antiparticle-state and I will refer to the multiplicity of those states (the collection for all spin $s$ and momenta $p$) as the $\bar{\nu}$-antiparticle. We can also say this differently as "$\bar{\nu}$" is the antiparticle of "$\nu$".

## 5.3 General states

I've already specified the single-particle states. All of the above generalizes in two directions. First, the Lagrangian may contain several different fields, and for each of these there is a collection of annihilation and creation operators and associated single-particle and single-antiparticle states. Second, for each field, more states can be created than just the single-particle states. The most general case is the Fock space, consisting of linear combinations of multi-particle states (I include here, states like $|\nu, \bar{nu}, s_1, s_2, p_1, p_2\rangle$ etc.). For multiple fields, all linear combinations of the above are permitted.

One notational complexity here, is that there is some freedom in the Lagrangian for how we label fields. So, for example, we could have written the traditional Dirac electron Lagrangian out of fields $\psi_L$ and $\psi_R$ rather than $\psi$ and following the naming conventions above, we'd say that these two fields correspond to particles $e^-_L$ and $e^-_R$ (and the corresponding antiparticles). However, for the usual massive Dirac theory, the E-L equations lead to relationships between the $e^-_L$ and $e^-_R$ creation operators, and in the end we combine things into a single creation operator called the $e^-$ creation operator. The point is, that once there are relations between the creation operators of a theory, there can be some resultant ambiguities about what we mean by a particle or antiparticle. Less ambiguous are the single-particle states since these are defined as the result of acting on the vacuum with a specific well-defined operator.

## 5.4 Wavefunctions

In the Schrodinger description of non-relativistic quantum mechanics, the fundamental quantity is a wavefunction, usually described as $\psi(x)$. The symbol $\psi$ in this context is **NOT** directly related to the field $\psi$ so for now, I'll use the symbol $\chi$ to refer to Schrodinger-style wavefunctions.

Mathematically speaking, a wavefunction is the representation of a state. The connection between Schrodinger's formalism and Heisenberg's is $\chi(x) = \langle x|\psi\rangle$ or $\tilde{\chi}(p) = \langle p|\psi\rangle$. The states $|x\rangle$ are eigenstates of the position operator and the states $|p\rangle$ are eigenstates of the momentum operator. We say that $\chi(x)$ is a position wavefunction representation of the state and that $\tilde{\chi}(p)$ is a momentum wavefunction representation of the state.

When we refer to the spinors $v$ and $p$ as antiparticles or states, what we really mean is that they are Schrodinger-style wavefunctions. The question, is how can we connect those to the Heisenberg states that we've been discussing and in turn, how are those related to particles. Start by considering a single-antiparticle state $|\bar{\nu}, p, s\rangle$. We want $v^s(p)$, for example, to be a particular wavefunction representation of that single-antiparticle state. So, the question is "what representation?" If we were following the example of Schrodinger wavefunctions, we would look for a complete set of orthonormal states that can be used to create the desired wavefunction representation. Here's an ansatz.

$$\langle 0|\psi(x) + \psi^*(x)|\bar{\nu}, p, s\rangle = v^{s\dagger}(p)e^{-ipx}. \tag{6}$$

We get this by expanding $\psi^\dagger$ and $\psi$ in terms of annihilation and creation operators (or alternatively, by applying the field to the vacuum and then expanding in single-antiparticle states). The same kind of manipulations can be used to interpret $u$ as a representation of single-particle states.

I have to admit that although this seems like a reasonable interpretation of $u^s(p)e^{-ipx}$ and $v^s(p)e^{ipx}$ as wavefunctions or representations of states, I really don't find it easy to apply this interpretation to the understanding of phenomena like the see-saw effect or even Majorana particles.

# 6 A single Majorana theory

Remain in the Weyl representation. A Majorana spinor is defined as a Dirac spinor, $\psi$ constrained by[3]

$$-i\gamma_2\psi^* = \psi. \tag{7}$$

We will associate this field with a particle $\nu$ and its antiparticle $\bar{\nu}$. The term on the left of the equation is known as the charge-conjugate of $\psi$ and is denoted by the symbol $\psi^C \equiv -i\gamma_2\psi^*$. The constraint expressed in terms of the up and down components of $\psi$ becomes one of the two following equivalent equations:

$$\begin{aligned} -i\sigma_2\psi^{*D} &= \psi^U, \\ i\sigma_2\psi^{*U} &= \psi^D. \end{aligned} \tag{8}$$

It should be mentioned here that there is no guarantee that this kind of a constraint could be satisfied by any solution of the Dirac equation. However, this particular constraint does have solutions.

Apply the constraint equations to the component Dirac equations of eq. (5) to get two equivalent equations,

$$\begin{aligned} i\partial_\mu\sigma^\mu\psi^D + im\sigma_2\psi^{*D} &= 0, \\ i\partial_\mu\bar{\sigma}^\mu\psi^U - im\sigma_2\psi^{*U} &= 0. \end{aligned} \tag{9}$$

[3] There is a bit of imprecision in the use of the asterisk($^*$) for linear operators. Strictly speaking, we should only use the dagger ($^\dagger$) to denote the Hermitian conjugate. However, when we have a column vector of operators such as the field $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix}$ and we want to indicate that the column should be transposed into a row, simultaneously performing the Hermitian conjugate on each element of the vector, then we write this as $\psi^\dagger$. If, instead, we want to conjugate each element but without taking the transpose, we write $\psi^*$. It is generally clear from context whether the asterisk is meant to represent the Hermitian conjugate of an operator, or the complex conjugate of a number.

It's important to recognize that although these equations are obtained by imposing a constraint on the Dirac equation, we could equally well think of either (equivalent) equation as standing on its own, independent of the Dirac equation from which it was derived. In that view, we refer to the mass term as a "Majorana mass".

Return to the Majorana constraint of eq. (7) and apply it to the solution of the Dirac equation given by eq. (3) . Separately equate the coefficients of $e^{ipx}$ or $e^{-ipx}$ to obtain

$$\sum_{s=1,2} a_p^s u^s(p) - i\gamma_2 b_p^s v^{*s}(p) = 0. \tag{10}$$

We will refer to the $a_p^s$'s as $\nu$ annihilation operators, and $b_p^s$'s as $\bar{\nu}$ annihilation operators. We can now infer from eq. (4) that

$$a_p^{1\dagger} = -b_p^{2\dagger}, a_p^{2\dagger} = b_p^{1\dagger}. \tag{11}$$

Since the solutions of eq (4) are only for momenta in the $y$-direction, the above equalities have been shown only for those momenta. However, the equalities generalize to all momenta. Now we can write a relationship between single-particle and single-antiparticle states by applying the above creation operators to the vacuum. We obtain

$$|\bar{\nu}, p, s\rangle = i\sigma_2^{ss'}|\nu, p, s'\rangle \tag{12}$$

**Notice that the Majorana constraint implies that the antiparticles are the same as the particles.**[4]

# 7 Lagrangian

## 7.1 Canonical forms and normalization

This section was added after I had run into some normalization difficulties with an approach based on a Lagrangian whose 4-spinor components are constrained by the Majorana condition. Those constraints are generically *holonomic* constraints for the variables in the Lagrangian, and require careful treatment in order to obtain the correct equations of motion and (anti-) commutation relations. To avoid those complications, this section describes unconstrained Lagrangians and their solutions in terms of annihilation and

[4]This is even more evident if we transform to the Majorana representation. In that representation, the self-charge-conjugation condition becomes a reality condition. The self-conjugacy of $\psi$ then directly leads to an equality between particles and anti-particles.

creation operators. We refer to these as *canonical* forms. Later on, as we introduce mixtures of fields, it will be important to force these into one of the canonical forms, so that we can derive correctly the physical consequences of the theory.

There are two distinct ways in which we use the Lagrangian to extract physics. One way, is to obtain the equations of motion. The other, which we call *quantization* is to obtain commutation (or anti-commutation) relations for the field operators. The transition from Lagrangian to scattering matrix is well-documented, but requires great care with signs, factors of 2, and so on. In order to avoid re-inventing the wheel each time a new theory is explored, it's useful to convert the Lagrangian into a canonical form whose behavior has been previously analyzed. The canonical forms of interest here are

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} &= i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi, \\ \mathcal{L}_{\text{Majorana}} &= i\nu^\dagger\bar{\sigma}^\mu\partial_\mu\nu + i\frac{m}{2}(\nu^T\sigma_2\nu - \nu^\dagger\sigma_2\nu^*),\end{aligned} \tag{13}$$

where $m$ is positive, $\psi$ is a 4-spinor fermion, and $\nu$ is a 2-spinor fermion. The notation $\bar{\phi}$ for any field $\phi$, is taken to mean $\phi^\dagger\gamma^0$, and the notation $a\!\!\!/$ for any vector $a$ is taken to mean $\gamma^\mu a_\mu$. Furthermore, in the canonical forms, there are no constraints.[5] When, later in the text, we introduce various qualifiers such as prime superscripts, for the $\psi$ and $\nu$ fields, these qualifiers will be implicitly inherited by the the components of those fields. So, for example, if we were to generically write $\psi = \begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$, then $\psi' = \begin{pmatrix}\psi'_A \\ \psi'_B\end{pmatrix}$ and similarly the subscripts $A$ and $B$ will be inherited by the components of their respective 2-spinors. Also, notice that there is an equivalent Majorana Lagrangian form where the first term is $i\nu^\dagger\sigma_\mu\partial_\mu\nu$.

The Euler-Lagrange equations lead to the equation of motions

$$\begin{aligned}i\partial\!\!\!/\psi - m\psi &= 0 \\ i\bar{\sigma}^\mu\partial_\mu\nu - im\sigma_2\nu^* &= 0.\end{aligned} \tag{14}$$

These are the Dirac equation (eq. (1)) and Majorana equations (eqs. (9) ) shown in the previous sections.

[5] Also note that these Lagrangians are real. When proving that, don't forget that the fields anticommute and that for Grassman variables, the complex conjugate of $\alpha\beta$ is $\beta^*\alpha^*$). The mass terms for the Majorana Lagrangians would be 0 if the fields didn't anti-commute.

The quantization conditions[6] are

$$\{\psi^{\alpha}(t,\mathbf{x}),\psi^{\dagger\beta}(t,\mathbf{y})\} = \delta^{\alpha\beta}\delta^{3}(\mathbf{x}\text{-}\mathbf{y}), \tag{15}$$

$$\{\nu^{\alpha}(t,\mathbf{x}),\nu^{\dagger\beta}(t,\mathbf{y})\} = \delta^{\alpha\beta}\delta^{3}(\mathbf{x}\text{-}\mathbf{y}). \tag{16}$$

All other anti-commutators between equal-time field components are zero.

With the canonical Dirac Lagrangian along with the resulting quantization conditions, with the canonical Dirac solution $\psi$ of eq. (3), and with the choices of $u$ and $v$ discussed in Appendix A, the operators $a_p^s$ and $b_p^s$ can be shown to satisfy the standard annihilation/creation anti-commutation relations.

Furthermore, if we impose the Majorana constraint of eq. (7) on the canonical solutions $\psi$, and if we also impose the relationship of eq. (11) between the operators $a_p^s$ and $b_p^s$ then the $a_p^s$ operators continue to satisfy the standard annihilation/creation anti-commutation relations.

### 7.2 The kinetic terms of the general weak-interaction neutrino fields

The general Lagrangian to be examined in these notes, is

$$\mathcal{L}_K = i\nu_L^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\nu_L + i\nu_R^{\dagger}\sigma^{\mu}\partial_{\mu}\nu_R - m(\nu_L^{\dagger}\nu_R + \nu_R^{\dagger}\nu_L) - i\frac{M}{2}(\nu_R^{T}\sigma_2\nu_R - \nu_R^{\dagger}\sigma_2\nu_R^{*}). \tag{17}$$

where $\nu_L$ and $\nu_R$ are two unconstrained fermion (i.e. anticommuting) fields, each with two components. The Lagrangian is invariant under Lorentz transformations which transform $\nu_L$ and $\nu_R$ as left- and right- chiral spinors.

## 8 Equations of Motion

The equations of motion are obtained by finding fields that solve

$$\delta\mathcal{L}_K = 0.$$

One common procedure which we'll adopt, is to define two 4-spinors as

$$\psi_A = \begin{pmatrix} \nu_L \\ i\sigma_2\nu_L^{*} \end{pmatrix}, \psi_B = \begin{pmatrix} -i\sigma_2\nu_R^{*} \\ \nu_R \end{pmatrix}. \tag{18}$$

[6]The quantization conditions are set by anti-commutation conditions between the independent variables and their conjugate momenta

In other treatments such as Schwartz, the subscripts $A$ and $B$ are replaced by $L$ and $R$. That notation has the advantage of reminding us how the spinors are constructed, but the disadvantage (more serious, in my opinion) of potentially misleading the reader into thinking that these 4-spinors have definite chirality. With these definitions, we can easily verify (by brute force expansion) that the equations of motion can be written as

$$i\begin{pmatrix} \partial\!\!\!/ & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} - \begin{pmatrix} 0 & m \\ m & M \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0. \tag{19}$$

The entries of the mass matrix are all implicitly 4 x 4 matrices. For example, the entry "m" is meant to denote $m\mathbf{I}_4$. Another notational simplification will be to define the 8-spinor $\psi \equiv \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$.

The resultant equations of motion are not in canonical form, so the next steps are to find a new basis of fields which convert the equations to canonical form. We will end up with two separate Majorana fields with separate masses, and then we will rewrite the original neutrino fields in terms of those Majorana fields.

Find the eigenvalues and eigenvalues[7] of the mass matrix $\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$, to construct a similarity transformation matrix $\mathbf{S} = \begin{pmatrix} S_{LL}\mathbf{I}_{4x4} & S_{LR}\mathbf{I}_{4x4} \\ S_{RL}\mathbf{I}_{4x4} & S_{RR}\mathbf{I}_{4x4} \end{pmatrix}$, with the property that $\mathbf{S}\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}\mathbf{S}^{-1} = \begin{pmatrix} m'' & 0 \\ 0 & M'' \end{pmatrix}$.

$$\begin{aligned} m'' &= -\frac{\sqrt{M^2+4m^2}-M}{2} \\ M'' &= \frac{\sqrt{M^2+4m^2}+M}{2} \\ \begin{pmatrix} S_{LL} & S_{LR} \\ S_{RL} & S_{RR} \end{pmatrix} &= \begin{pmatrix} 1 & \frac{M-\sqrt{M^2+4m^2}}{2m} \\ 1 & \frac{\sqrt{M^2+4m^2}+M}{2m} \end{pmatrix} \end{aligned} \tag{20}$$

$m''$ turns out to be negative (which is not canonical), but that will be dealt with later. Define $\psi'' = \mathbf{S}\psi$, so

$$\psi'' \equiv \begin{pmatrix} \psi''_A \\ \psi''_B \end{pmatrix} = \begin{pmatrix} S_{LL}\psi_A + S_{LR}\psi_B \\ S_{RL}\psi_A + S_{RR}\psi_B \end{pmatrix}. \tag{21}$$

[7]I found it very helpful to do the algebraic manipulations using the program MAXIMA.

Rewrite the equations of motion, eq. (19) as

$$
\begin{aligned}
0 &= i\mathbf{S}^{-1}\mathbf{S}\begin{pmatrix} \partial\!\!\!/ & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix}\mathbf{S}^{-1}\mathbf{S}\psi - \mathbf{S}^{-1}\mathbf{S}\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}\mathbf{S}^{-1}\mathbf{S}\psi \\
&= i\mathbf{S}^{-1}\begin{pmatrix} \partial\!\!\!/ & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix}\mathbf{S}\psi - \mathbf{S}^{-1}\begin{pmatrix} m'' & 0 \\ 0 & M'' \end{pmatrix}\mathbf{S}\psi \\
&= \mathbf{S}^{-1}\begin{pmatrix} i\partial\!\!\!/ & 0 \\ 0 & i\partial\!\!\!/ \end{pmatrix}\psi'' - \mathbf{S}^{-1}\begin{pmatrix} m'' & 0 \\ 0 & M'' \end{pmatrix}\psi''.
\end{aligned}
\tag{22}
$$

In the second line, we have used that $\begin{pmatrix} \partial\!\!\!/ & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix}$ is proportional to the identity so it commutes with $\mathbf{S}$. Now multiply on the left by $\mathbf{S}$, and expand into the "up" and "down" components of $\psi''$ to get

$$
\begin{aligned}
i\partial\!\!\!/\psi''_A - m''\psi''_A = 0 \\
i\partial\!\!\!/\psi''_B - M''\psi''_B = 0
\end{aligned}
\tag{23}
$$

These equations are almost in canonical form (as will be seen shortly) except that $m''$ is negative (we'll take care of that later by a rescaling of $\psi_A$ by $-i\gamma_5$). **However**, although the equations are in (almost) canonical form, the Lagrangian is not. So before the final set of transformations to canonical form, we will need to inspect the Lagrangian and rescale (aka renormalize) the fields. The reason for all this, will turn out to be a consequence of the fact that $\mathbf{S}$ is not unitary.

We start the analysis of the Lagrangian by rewriting $\mathcal{L}_K$ as

$$
\mathcal{L}_K = \frac{1}{2}\left(i\bar{\psi}\begin{pmatrix} \partial\!\!\!/ & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix}\psi - \bar{\psi}\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}\psi\right),
$$

where, as a reminder $\psi$ is defined to be $\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$ and $\psi_{A,B}$ are defined in eqs. (18). Then, similarly to what we did in eq. (22),

$$
\begin{aligned}
\mathcal{L}_K &= \frac{1}{2}\left(i\bar{\psi}\mathbf{S}^{-1}\mathbf{S}\begin{pmatrix} \partial\!\!\!/ & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix}\mathbf{S}^{-1}\mathbf{S}\psi - \bar{\psi}\mathbf{S}^{-1}\mathbf{S}\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}\mathbf{S}^{-1}\mathbf{S}\psi\right) \\
&= \frac{1}{2}\left(i\bar{\psi}\mathbf{S}^{-1}\begin{pmatrix} \partial\!\!\!/ & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix}\mathbf{S}\psi - \bar{\psi}\mathbf{S}^{-1}\begin{pmatrix} m'' & 0 \\ 0 & M'' \end{pmatrix}\mathbf{S}\psi\right) \\
&= \frac{1}{2}\left(i\bar{\psi}''\left(\mathbf{S}^{-1}\right)^{\dagger}\mathbf{S}^{-1}\begin{pmatrix} \partial\!\!\!/ & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix}\psi'' - \bar{\psi}''\left(\mathbf{S}^{-1}\right)^{\dagger}\mathbf{S}^{-1}\begin{pmatrix} m'' & 0 \\ 0 & M'' \end{pmatrix}\psi''\right).
\end{aligned}
\tag{24}
$$

Using the values of $\mathbf{S}$ and $\mathbf{S}^{-1}$ obtained in eq. (20), we get

$$\left(\mathbf{S}^{-1}\right)^{\dagger}\mathbf{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1+\frac{M}{\sqrt{4m^2+M^2}} & 0 \\ 0 & 1-\frac{M}{\sqrt{4m^2+M^2}} \end{pmatrix}.$$

Substitute this into eq. (24) and expand $\psi''$ into its components to get

$$\begin{aligned}\mathcal{L}_K = {} & i\left(\frac{1+\frac{M}{\sqrt{4m^2+M^2}}}{4}\right)\bar{\psi}''_A \partial\!\!\!/ \psi''_A + i\left(\frac{1-\frac{M}{\sqrt{4m^2+M^2}}}{4}\right)\bar{\psi}''_B \partial\!\!\!/ \psi''_B \\ & + \frac{m^2}{2\sqrt{4m^2+M^2}}\bar{\psi}''_A\psi''_A - \frac{m^2}{2\sqrt{4m^2+M^2}}\bar{\psi}''_B\psi''_B\end{aligned} \tag{25}$$

Notice that $\psi''_A = \begin{pmatrix}\psi''^U_A \\ i\sigma_2\psi''^{U*}_A\end{pmatrix}$ and $\psi''_B = \begin{pmatrix}\psi''^U_B \\ i\sigma_2\psi''^{U*}_B\end{pmatrix}$. These equalities follow from eq. (18) where $\psi^D_{A,B} = i\sigma_2\psi^{U*}_{A,B}$. This fact can be used to expand $\mathcal{L}_K$ into terms involving only $\psi''^U_A$ and $\psi''^U_B$.

$$\begin{aligned}\mathcal{L}_K = {} & i\left(\frac{1+\frac{M}{\sqrt{4m^2+M^2}}}{2}\right)\psi''^{U\dagger}_A\bar{\sigma}^\mu\partial_\mu\psi''^U_A + i\left(\frac{1-\frac{M}{\sqrt{4m^2+M^2}}}{2}\right)\psi''^{U\dagger}_B\bar{\sigma}^\mu\partial_\mu\psi''^U_B \\ & -i\frac{m^2}{2\sqrt{4m^2+M^2}}(\psi''^{UT}_A\sigma_2\psi''^U_A - \psi''^{U\dagger}_A\sigma_2\psi''^{U*}_A) + i\frac{m^2}{2\sqrt{4m^2+M^2}}(\psi''^{UT}_B\sigma_2\psi''^U_B - \psi''^{U\dagger}_B\sigma_2\psi''^{U*}_B)\end{aligned} \tag{26}$$

This equation can be converted to canonical form by redefining the fields and masses so that

$$\begin{aligned}\psi'_A &= -\gamma_5\sqrt{\frac{1+\frac{M}{\sqrt{4m^2+M^2}}}{2}}\psi''_A \\ \psi'_B &= \sqrt{\frac{1-\frac{M}{\sqrt{4m^2+M^2}}}{2}}\psi''_B \\ m' &= \frac{m^2}{\sqrt{4m^2+M^2}}\left(\frac{1+\frac{M}{\sqrt{4m^2+M^2}}}{2}\right)^{-1} \\ M' &= \frac{m^2}{\sqrt{4m^2+M^2}}\left(\frac{1-\frac{M}{\sqrt{4m^2+M^2}}}{2}\right)^{-1}\end{aligned} \tag{27}$$

Then

$$\begin{aligned}\mathcal{L}_K = {} & i\psi'^{U\dagger}_A\bar{\sigma}^\mu\partial_\mu\psi'^U_A + i\psi'^{U\dagger}_B\bar{\sigma}^\mu\partial_\mu\psi'^U_B \\ & + i\frac{m'}{2}(\psi'^{UT}_A\sigma_2\psi'^U_A - \psi'^{U\dagger}_A\sigma_2\psi'^{U*}_A) + i\frac{M'}{2}(\psi'^{UT}_B\sigma_2\psi'^U_B - \psi'^{U\dagger}_B\sigma_2\psi'^{U*}_B)\end{aligned} \tag{28}$$

which is in the Majorana canonical form as promised.

In summary, we have found that our theory describes two Majorana fermions with (Majorana) masses $m'$ and $M'$. A popular hypothesis is that $m$ is on the order of the electroweak scale $\approx 100$ GeV and that $M$ is on the order of the Planck scale $\approx 10^{19}$GeV. Then $m \ll M$ and we can expand the terms as

$$m' \approx \frac{m^2}{M}, \quad M' \approx M. \tag{29}$$

# 9 Relationship of the massive Majorana particles to the neutrino interactions with weak vector mesons etc.

The fields which appear in the original interaction terms with weak vector mesons (or in the 4-fermi theory) are $\nu_L$ and $\bar{\nu}_L$. More recently, after the discovery of neutrino masses, there are additional fields $\nu_R$ and $\bar{\nu_R}$. However, none of those fields can be interpreted as particle fields. A particle field is associated with particles of definite mass. The $\nu_L$ and $\nu_R$ fields are mixtures of the definite-mass fields $\psi_A'^U$ and $\psi_B'^U$ which we've derived above. In what follows, we write out the mixture in a few limiting cases.

It is helpful to gather the various defining transformations in one place. From eq.(18) we have

$$\psi_A = \begin{pmatrix} \nu_L \\ i\sigma_2\nu_L^* \end{pmatrix}, \psi_B = \begin{pmatrix} -i\sigma_2\nu_R^* \\ \nu_R \end{pmatrix}. \tag{30}$$

From eq. (20) we have

$$\mathbf{S} = \begin{pmatrix} S_{LL} & S_{LR} \\ S_{RL} & S_{RR} \end{pmatrix} = \begin{pmatrix} 1 & \frac{M-\sqrt{M^2+4m^2}}{2m} \\ 1 & \frac{\sqrt{M^2+4m^2}+M}{2m} \end{pmatrix} \tag{31}$$

and from eq. (21)

$$\psi'' \equiv \mathbf{S} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}. \tag{32}$$

In the above equations, we have suppressed the implicit factor of the identity matrix $\mathbf{I}_{4x4}$ in each of the matrix blocks. From eq. (27) we have

$$\begin{pmatrix} \psi''_A \\ \psi''_B \end{pmatrix} = \begin{pmatrix} -\gamma_5\sqrt{\frac{2}{1+\frac{M}{\sqrt{4m^2+M^2}}}}\psi'_A \\ \sqrt{\frac{2}{1-\frac{M}{\sqrt{4m^2+M^2}}}}\psi'_B \end{pmatrix} \tag{33}$$

These equations lead to the following expressions for $\nu_R$ and $\nu_L$ in terms of the Majorana fields:

$$\nu_L = \psi_A^U \,, \nu_R = \psi_B^D, \tag{34}$$

where

$$\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \mathbf{S}^{-1} \begin{pmatrix} -\gamma_5 \sqrt{\frac{2}{1+\frac{M}{\sqrt{4m^2+M^2}}}}\psi'_A \\ \sqrt{\frac{2}{1-\frac{M}{\sqrt{4m^2+M^2}}}}\psi'_B \end{pmatrix} \tag{35}$$

and we can compute

$$\mathbf{S}^{-1} = \frac{1}{\sqrt{4m^2+M^2}} \begin{pmatrix} \frac{\sqrt{4m^2+M^2}+M}{2} & \frac{\sqrt{4m^2+M^2}-M}{2} \\ -m & m \end{pmatrix}. \tag{36}$$

Altogether

$$\begin{aligned}
\nu_L &= \frac{1}{\sqrt{m'+M'}} \left( \sqrt{M'}\psi'^U_A + \sqrt{m'}\psi'^U_B \right) \\
&= \frac{1}{(4m^2+M^2)^{\frac{1}{4}}} \left( \sqrt{\frac{\sqrt{4m^2+M^2}+M}{2}}\psi'^U_A + \sqrt{\frac{\sqrt{4m^2+M^2}-M}{2}}\psi'^U_B \right) \\
\nu_R &= \frac{1}{\sqrt{m'+M'}} \left( \sqrt{m'}\psi'^U_A + \sqrt{M'}\psi'^U_B \right) \\
&= \frac{1}{(4m^2+M^2)^{\frac{1}{4}}} \left( \sqrt{\frac{\sqrt{4m^2+M^2}-M}{2}}\psi'^D_A + \sqrt{\frac{\sqrt{4m^2+M^2}+M}{2}}\psi'^D_B \right)
\end{aligned} \tag{37}$$

These equations express the usual neutrino fields (appearing in weak interaction terms) as superpositions of the massive Majorana fields $\psi_A^U$ and $\psi_B^D$.

## 9.1 $m << M$

Expand the square-root in eqs. (37) and use eq. (30) to get $\nu_L$ and $\nu_R$.

$$\begin{aligned}
\nu_L &= \left(1 - \frac{m^2}{2M^2}\right)\psi'^U_A - i\frac{m}{M}\left(1 - \frac{3m^2}{2M^2}\right)\sigma_2\psi'^{*D}_B + ..., \\
\nu_R &= \left(1 - \frac{m^2}{2M^2}\right)\psi'^D_B + i\frac{m}{M}\left(1 - \frac{3m^2}{2M^2}\right)\sigma_2\psi'^{*U}_A + ....
\end{aligned} \tag{38}$$

### 9.2 M=0

This should lead to the interpretation of neutrinos as Dirac particles of the form $\psi = \begin{pmatrix} \nu_L \\ \nu_R \end{pmatrix}$. Substituting $M = 0$ into eq. (37), we obtain

$$\begin{aligned} \nu_L &= \frac{1}{2}\left(\psi'^U_A + \psi'^U_B,\right) \\ \nu_R &= \frac{1}{2}\left(\psi'^D_A + \psi'^D_B,\right) \end{aligned} \tag{39}$$

or

$$\psi = \frac{1}{2}\left(\psi'_A + \psi'_B\right). \tag{40}$$

Both $\psi'_A$ and $\gamma_5\psi'_B$ satisfy the Majorana conditions and therefore are Majorana fields. This demonstrates a point made at the beginning: Massive Dirac fields can be constructed out of two Majorana fields, and therefore a Dirac field theory is simply a special case of a field theory of a pair of Majorana fields.

## 10 Particles instead of fields

This section starts with some elaboration of parts of section (5). Assume that there is a unique vacuum $|0\rangle$ which is invariant under the symmetries to be discussed in what follows. In the standard model, there is actually a family of vacua that transform into one another via a subgroup of SU(2) x U(1), and symmetry is broken by the selection of one of those vacua. However, we can ignore that for now.

We define the following (normalized) single-particle states using the canonical solutions for $\psi'_A$ and $\psi'_B$ – given respectively in terms of annihilation and creation operators subscripted by $A$ and $B$. So

$$|A, p, s\rangle = \frac{1}{\sqrt{2\omega^{m'}_{\mathbf{p}}}} a^{\dagger' s}_{Ap}|0\rangle\,,\ \ |B, p, s\rangle = \frac{1}{\sqrt{2\omega^{M'}_{\mathbf{p}}}} a^{\dagger' s}_{Bp}|0\rangle \tag{41}$$

and (normalized) single-antiparticle states:

$$|\bar{A}, p, s\rangle = \frac{1}{\sqrt{2\omega^{m'}_{\mathbf{p}}}} b^{\dagger' s}_{Ap}|0\rangle\,,\ \ |\bar{B}, p, s\rangle = \frac{1}{\sqrt{2\omega^{M'}_{\mathbf{p}}}} b^{\dagger' s}_{Bp}|0\rangle. \tag{42}$$

The momentum is on the mass-shell. The term **particle** is used to denote[8] a set of states that include $|A, p, s\rangle = \frac{1}{\sqrt{2\omega^{m'}_{\mathbf{p}}}} a^{\dagger' s}_{Ap}|0\rangle$ and mix only among themselves under Poincaré transformations. Similarly with the term **antiparticle**.

[8]See Schwartz p. 110

When we apply Majorana constraints, we find from eqs. (11), the relationships between the states:

$$\begin{aligned} |\bar{A}, p, s\rangle &= -i\sigma_2^{ss'}|A, p, s'\rangle \\ |\bar{B}, p, s\rangle &= i\sigma_2^{ss'}|B, p, s'\rangle \end{aligned} \tag{43}$$

That is, the antiparticle states are particle states. Notice that the $B$ states transform with opposite signs to the $A$ states. This is because the $A$ particles are defined by the creation operators for the $\psi'_A$ field, which was obtained from the $\psi''_A$ field that satisfies $\psi''^C_A = -i\gamma_2\psi''^*_A = \psi''_A$, by multiplying by a term proportional to $\gamma_5$. In other words, $\psi'^C_A = i\gamma_2\psi'^*_A$.

Next, derive wavefunctions $u$ and $v$ as follows (see section (5) for more information). First define

$$\Phi(x) = \psi'^*_A(x) + \psi'_A(x) + \psi'^*_B(x) + \psi'_B(x).$$

Then

$$\begin{aligned} e^{-ipx}u'^s_A(m', p) &= \langle 0|\Phi(x)|A, p, s\rangle \\ e^{-ipx}v'^{s*}_A(m', p) &= \langle 0|\Phi(x)|\bar{A}, p, s\rangle \\ e^{-ipx}u'^s_B(M', p) &= \langle 0|\Phi(x)|B, p, s\rangle \\ e^{-ipx}v'^{s*}_B(M', p) &= \langle 0|\Phi(x)|\bar{B}, p, s\rangle. \end{aligned} \tag{44}$$

To illustrate how we can work with these wavefunctions, define the charge-conjugation operation on the single-particle states by

$$\begin{aligned} |\bar{A}, p, s\rangle &\to -i\sigma_2^{ss'}|A, p, s'\rangle \\ |\bar{B}, p, s\rangle &\to i\sigma_2^{ss'}|B, p, s'\rangle. \end{aligned} \tag{45}$$

By applying this to the single-particle states in eq. (44), we find that

$$\begin{aligned} v'^{s*}_A &\to -i\sigma_2^{ss'}u'_{As'} \\ v'^{s*}_B &\to i\sigma_2^{ss'}u'_{Bs'}, \end{aligned} \tag{46}$$

which is the same as

$$\begin{aligned} v'^s_A &\to -i\gamma_2 v'^{s*}_A \\ v'^s_B &\to i\gamma_2 v'^{s*}_B \end{aligned} \tag{47}$$

This illustrates the connection between wavefunction charge-conjugation and particle charge-conjugation (which derives from field charge-conjugation). Overall, I don't find the wavefunction approach especially helpful since its connection to either particles or fields – both of which I have a better feeling for – is a bit indirect.

Now let's return to the single-particle picture. In the previous section on the neutrino and Majorana particles, we ended up with eqs. (37) which relates the neutrino fields $\nu_R$ and $\nu_L$, to the Majorana fields $\psi'_A$ and $\psi'_B$. The Majorana fields are associated with the elementary particles $|A, p, s\rangle$ and $|B, p, s\rangle$ which are also their own antiparticles. The neutrino fields of the last section are **NOT** associated with elementary particles. We could rewrite the Lagrangian and therefore the entire theory, purely in terms of the Majorana fields and their associated Majorana particles.

**However, there's one more step required in order to connect this formalism to experimental observations of particles.** We identify the light Majorana particle (associated with field $\psi'_A$) as *the neutrino*! So, even though the neutrino-field isn't associated with an elementary particle, the light Majorana field is, and we call that particle the neutrino.

Why the odd distinction between the name of the field ($\psi'_A$)and the name of the particle ($\nu$)? That comes about from consideration of the neutrino interactions with electro-weak vector bosons and Higgs bosons. Those interactions are described in terms of fields rather than the particles. The interaction of the Majorana particles with the $W^{\pm}$ and $Z$ bosons, for example, is specified by the Lagrangian term involving the $\nu_L$ field (which is a particular sum of the Majorana fields) and those bosons. Similarly, the sterility of $\nu_R$ becomes a statement about the non-interactivity of a different sum of the 2 Majorana fields.

The effect is exactly of the same nature as one would have if, for example, one were to examine the interactions of an electromagnetic field with two charged elementary particles of different flavors. The interaction term would appear as a sum. When translated into Feynman diagrams, the effect is to have graphs with two kinds of vertices – one involving the bosons and $\psi'_A$ and the other involving the bosons and $\psi'_B$. Their relative coupling constants are determined by the coefficients in the linear decomposition of the neutrino field.

So, looking at the first of eqs. (37), we see that the coupling of the bosons to $|A, p, s\rangle$ (which we call 'the neutrino' or even more precisely $|\nu_L, p, s\rangle$) is proportional to $\frac{\sqrt{M'}}{\sqrt{m'+M'}}$ and the coupling to $|B, p, s\rangle$ (which we call the sterile neutrino and which we write as $|\nu_R, p, s\rangle$)is proportional to $\frac{\sqrt{m'}}{\sqrt{m'+M'}}$.

# A Solutions of the Dirac equation for momentum in the $y$ direction

The expression of $u$ and $v$ spinors for general momenta is given by Schwartz on p. 191, and requires calculating the squareroots of matrices.

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix} \tag{48}$$

where $\sigma$ and $\bar{\sigma}$ are defined just after eq. (4), and

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{49}$$

There is a trick for calculating, for example, $\sqrt{\boldsymbol{\sigma}\cdot p}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Notice that $\boldsymbol{\sigma}\cdot p$ can be diagonalized by a similarity transformation $\mathbf{S}$. That is, $\mathbf{S}(\boldsymbol{\sigma}\cdot p)\mathbf{S}^{-1} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$. Then notice that if $\mathbf{X}$ is a matrix, and $f(\mathbf{X})$ is a polynomial function of that matrix, we have $\mathbf{S}f(\mathbf{X})\mathbf{S}^{-1} = f(\mathbf{S}\mathbf{X}\mathbf{S}^{-1})$. By a Taylor expansion argument, we can apply this observation to $\boldsymbol{\sigma}\cdot p$.

$$\begin{aligned} \sqrt{\boldsymbol{\sigma}\cdot p}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \mathbf{S}^{-1}\mathbf{S}(\sqrt{\boldsymbol{\sigma}\cdot p})\mathbf{S}^{-1}\mathbf{S}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \mathbf{S}^{-1}(\sqrt{\mathbf{S}\boldsymbol{\sigma}\cdot p\mathbf{S}^{-1}})\mathbf{S}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \mathbf{S}^{-1}\sqrt{\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}}\mathbf{S}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \mathbf{S}^{-1}\begin{pmatrix} \sqrt{e_1} & 0 \\ 0 & \sqrt{e_2} \end{pmatrix}\mathbf{S}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{50}$$

Let's apply this to the case $p^\mu = (E, 0, p_y, 0)$. Then[9]

$$p\cdot\sigma = \begin{pmatrix} E & ip_y \\ -ip_y & E \end{pmatrix} \tag{51}$$

$\mathbf{S}$ can be constructed out of the eigenvectors of $p\cdot\sigma$ (for which MAXIMA is a great tool) and we get

$$\mathbf{S} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \mathbf{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{52}$$

[9]It's pretty easy to get messed up with summation conventions along with lowering-raising index conventions. I've tried to be self-consistent, but if I messed up, the worst would be a sign-reversal of the spacial momenta.

Then

$$\mathbf{S}^{-1} p \cdot \sigma \mathbf{S} = \begin{pmatrix} E - p_y & 0 \\ 0 & E + p_y \end{pmatrix} \tag{53}$$

and

$$\begin{aligned} \sqrt{p \cdot \sigma} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \mathbf{S}^{-1} \begin{pmatrix} \sqrt{E - p_y} & 0 \\ 0 & \sqrt{E + p_y} \end{pmatrix} \mathbf{S} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} i(\sqrt{E + p_y} - \sqrt{E - p_y}) \\ \sqrt{E + p_y} + \sqrt{E - p_y} \end{pmatrix} \end{aligned} \tag{54}$$

Similarly, other components of $u^s$ and $v^s$ can be calculated.