Unpacking Susskind's lecture on the Higgs phenomenon

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Susskind, in his lecture on the Higgs phenomenon, covers a number of topics from occasionally idiosyncratic viewpoints, and certainly eschewing the abstract mathematical approach that we've been taking. Here are a few items that caught my attention.

- Quantization of angular momentum
- Quantization of charge
- Meaning of the vacuum
- Mass is the frequency of chiral flips
- Condensates and the Higgs phenomenon

1 Quantization of angular momentum

Susskind says that the only two things we need to know about quantum mechanics are

- Angular momentum is quantized.
- The Heisenberg uncertainty principle – if you know that the system is at rest, then you don't know how fast it's moving and vice versa.

I want to concentrate for now on the angular momentum. You might wonder why Susskind highlighted quantization rather than Schrodinger's equation or Hilbert spaces etc. And then you might wonder "why angular momentum rather than plain old momentum?" First of all, Susskind, by discussing quanta, is highlighting the very first historical insight into quantum mechanics, and arguably the driving force behind the entire evolution of the subject matter. The mathematics is, in many ways, a by-product of the effort to explain both the cause and effect of quantization. I don't want to reconstruct the history of the subject, but instead will focus your attention on the plain old Schrodinger equation.

It is simplest to imagine the quantum mechanics of a 2D space rather than the real-world 3D space. I'm going to always take $\hbar = 1$. Remember that plain old momentum has to do with linear motion in space. For example, in the x-direction, the momentum operator is $-i\frac{\partial}{\partial x}$. This is the quantum analogue of the classical variable p_x. You might remember that the Schrodinger equation involves the action of the energy, or Hamiltonian, on a state. In 2D classical mechanics, the energy (for a freely moving particle) is

$$E = \frac{p_x^2 + p_y^2}{2m}.$$

So in QM, we have

$$H = \frac{\left(-i\frac{\partial}{\partial x}\right)^2 + \left(-i\frac{\partial}{\partial y}\right)^2}{2m}.$$

Also remember that plane waves are "eigenstates" of the momentum operator

$$-i\frac{\partial}{\partial x} e^{\pm i(k_x x + k_y y)} = \pm k_x e^{\pm i(k_x x + k_y y)},$$

and similarly with the y-momentum. Above, $\pm k_x$ is the eigenvalue. Then, since the Hamiltonian is proportional to the square of the momentum, the plane waves are also eigenstates of the Hamiltonian. It should be stated that plane waves, mathematically speaking, aren't actually states of the Hilbert space. They aren't normalizable. However, there are reasonable ways of extending the concept of Hilbert space to include plane waves. A similar thing is also true of some condensates which I'll consider later.

The reason I concentrated so much on linear momentum is that this is **NOT** quantized. Any value of k_x is permissible. You might recall that in a system with boundary conditions, the linear momentum is quantized but in "empty space" there aren't any boundary conditions of that kind.

However, we'll now see that angular momentum **is** quantized. In 2D, angular momentum has to do with the circular motion around a center point, measured by the angle θ. We describe the circularly-symmetric world with circular coordinates – in this case θ. The angular momentum operator is

$$-i\frac{d}{d\theta}$$

. The eigenstates satisfy

$$-i\frac{d}{d\theta}e^{\pm in(2\pi\theta)} = \pm n(2\pi)e^{\pm in(2\pi\theta)},$$

where n is an integer. The angular momentum eigenvalues are $\pm 2\pi n$. Why is n an integer? Because the wavefunction better be the same after a rotation of 2π as it was before rotation. We say that the coordinate is periodic. **So the angular momentum eigenvalues are quantized and we say $\pm n$ are the values of the quanta.**

What is the deep reason that angular momentum is quantized, but linear momentum isn't? It has to do with the fact that linear momentum is the operator generating transformations of the linear coordinates i.e.

$$\psi(x) \to \psi(x+\delta)$$

and that angular momentum is the operator generating rotations

$$\psi(\theta) \to \psi(\theta+\delta)$$

. We've encountered this concept when we were doing the fancy group theory stuff. Remember the Lie Algebra? Those were the generators of rotations in 3D. We mentioned that these were also known as the angular momentum operators. Specifically, in terms of operators on the Hilbert space, angular momentum $\mathbf{J}$, is the operator which generates rotations as $\mathbf{R}(\theta) = e^{-i\mathbf{J}\cdot\boldsymbol{\theta}}$.

The reason that the angular-momentum eigenvalues are quantized, is that rotations are a compact symmetry group (picture the group as closed and bounded). By contrast, the translation group is unbounded. You can translate by as much as you want and you don't end up coming back to yourself (unless you're doing general relativity but that's a story for another time). It's a feature of compact symmetry groups that the generators have discrete (quantized) eigenvalues.[1]

[1] By the way, remember I said that the rotation quanta were $\pm n$? We saw this came about because a rotation of 2π has to bring the wavefunction back to itself. But does it? In a quantum world, the wavefunction $-\psi(x)$ is the same as $\psi(x)$ because the phase can't be measured. We've encountered this before. It allowed us to consider wavefunctions that have to go around by 4π before they return to themselves (familiar to quaternion practitioners). We said this kind of thing isn't really a rotation, but rather an element of $SU(2)$. Anyway, if the condition is that $e^{i\pm 2n(2\pi)} = 1$ then the eigenvalues are $\pm\frac{n}{2}(2\pi)$ or said differently, the quanta are in units of $\frac{n}{2}$. And of course, that's what permits electrons to have spin $\frac{1}{2}$.

2 Quantization of charge

Back to Susskind. Even though he speaks of quantization as related to angular momentum, his real interest isn't angular momentum – which generates rotational symmetry – but some other generator of a different compact group (in particular, a compact subgroup of the group of weak and electroweak interactions $U(1) \times SU(2)$.) He steers clear, as much as possible, from the language of symmetry. Instead, he focuses on *charge*, which is a related concept. Let's make the connection. We visualize the field angular momentum as measuring the angular motion of the field. Susskind depicts this by drawing circles in his Mexican hat. He's only using the angular momentum idea to explain the key concept he's interested in, namely charge. Just like angular momentum has discrete eigenvalues roughly corresponding to angular speed, so does our field theory symmetry generator have discrete eigenvalues roughly corresponding to the tangent vector of some symmetry parameter. On the Mexican hat, that parameter is given by the angle around the center and Susskind calls[2] the generator eigenvalues ***Zilch***.[3] Since the symmetry generator has the same relationship to the angle, as momentum has to position, Susskind then applies the Heisenberg uncertainty principle to conclude that if the angle is known precisely, then the generator eigenvalue – i.e. Zilch – is totally unknown.

3 Meaning of the vacuum

All of us have been thoroughly indoctrinated in the idea that zero is a number on par with other numbers. This is a concept of mathematics. But it's worth remembering that zero was an abstraction added relatively recently to the whole idea of counting.

When we speak of the universe, the concept of 'nothing' is an abstraction not required by any experience we've had. Let's consider quantum mechanics. We have a collection of states that make up the Hilbert space and we have a metric. The physical states are regarded as the normalizable vectors in the Hilbert space. In particular, **these are all non-zero vectors in the Hilbert space**. Any vector other than 0 can be multiplied by a scalar so that the result has a unit norm.

[2]The terms Zilch and Ziggs are inventions of Susskind and probably won't be found anywhere else in the literature – although I haven't checked.

[3]The actual story of electromagnetic-charge quantization is much more subtle than the story of angular-momentum quantization, but it is true that one of the ingredients of that story, is the symmetry of a compact group.

In other words, in Quantum Mechanics, 0 is **NOT** a physical state. And as a consequence, there is no physical state that represents a universe that contains *nothing*. Within any quantum theory, there is a state that has a preferred status. This is the state which is an eigenvector of the Hamiltonian (energy operator) whose energy-eigenvalue is the lowest among all energy eigenvalues.

There's no mathematical reason why a theory should have only one lowest-energy eigenstate. But, for a long'ish period of time, it was regarded as an axiom of physics, that the lowest-eigenvalue energy eigenstate is unique. When that is true, then symmetries of motion – i.e., operators that commute with the Hamiltonian – preserve that unique state (i.e. it doesn't change). And that's a very useful starting point for many interesting effects.

We call that unique eigenstate 'the vacuum'. **IT DOES NOT REPRESENT '0'!** Unfortunately, in an abuse of notation, the vacuum is often described, in bra-ket notation, as $|0\rangle$, which tends to mislead everyone into thinking it means 'nothing'. So, one of the things Susskind does early on, is to posit the existence of a vacuum state between charged capacitors and which consists of an electric field pointing in some direction. That's not a problem! Like I said, an apparently empty universe is represented by a normalizable state, and that state is far from trivial. Different observables (represented by operators) have different expectation values in that vacuum state.

In modern times, there is one more (critical) generalization of this story. It is not longer regarded as axiomatic, that the vacuum be unique. We can study theories where there are many (even infinite) states of lowest energy. The Higgs phenomenon is a particular manifestation of this kind of thing.

It helps even more than before, to realize there's nothing 'empty' about a world whose vacuum state is degenerate.

4 Mass and chirality

Susskind interprets the massive Dirac equation as describing a time evolution in which the particle's chirality flips back and forth at a rate which is proportional to its mass. Or said differently, the mass is really a 'frequency of chiral-flipping'. I have to admit that this interpretation of the Dirac equation isn't one I've heard stated before. But it's not hard to see that it's a reasonable way of interpreting the Dirac equation.

To see this, we need to recall some notes called *Introducing the Dirac equation.* Here are some of the key equations from those notes. First recall that we had two 2-spinors that we called ψ_L and ψ_R and when expanded out, they obey Lorentz-invariant differential equations which, when restricted to the z-axis become

$$\begin{pmatrix} (i\partial_0 - i\partial_3)\psi_L^1(x) - (i\partial_1 + \partial_2)\psi_L^2(x) \\ (i\partial_0 + i\partial_3)\psi_L^2(x) - (i\partial_1 - \partial_2)\psi_L^1(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1}$$

$$\begin{pmatrix} (i\partial_0 + i\partial_3)\psi_R^1(x) + (i\partial_1 + \partial_2)\psi_R^2(x) \\ (i\partial_0 - i\partial_3)\psi_R^2(x) + (i\partial_1 - \partial_2)\psi_R^1(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2}$$

These are two **separate** equations for ψ_L and ψ_R. We say that the complete set of differential equations is *separable*. Up to now, $\psi_{R,L}$ have represented fields and we haven't talked about particles. There's a relationship, and for the remainder of this discussion, I'll sometimes refer to fields as describing particles. Crudely speaking, the fields are operators which, when they operate on the vacuum state (see the last section), result in particle states.

There is a nice condensed form of these equations which is known as the Dirac equation:

$$\begin{pmatrix} i(\mathbf{I}_2\partial_0 + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}) & 0 \\ 0 & i(\mathbf{I}_2\partial_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}) \end{pmatrix} \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix} = \mathbf{0}. \tag{3}$$

Each block of the matrix is 2 x 2 and each entry in the vector is a 2-spinor. The separability of the equations arises because the matrix is block-diagonal.

There are two related ways in which this equation looks different than the usual Dirac equation. First, there is no mass term. Second, the usual Dirac equation is expressed using 4x4 matrices known as the gamma matrices. It turns out that when there is no mass term, the usual Dirac equation becomes the above equation.

Although I don't want to talk yet about the complete solution to these equations, I do want to give a hint. It turns out that the solutions can be expressed as Fourier transforms of the form $\int d^3p\,(\chi_+(p)e^{ip\cdot x} + \chi_-(p)e^{-ip\cdot x})$ where $p = (\omega_0(\mathbf{p}), p^1, p^2, p^3)$ and $\omega_0(\mathbf{p}) = \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}$. In other words, the energy-momentum of the solutions, have the same relationship of energy to momentum (a.k.a. the *dispersion relation*) as particles without mass, and therefore travelling at the speed of light. We'll return to that shortly.

Now it's time to talk about our spinor notation. We use the subscripts 'L' and 'R' to describe a quantity called chirality. For those following the group-theoretic mathematics of the Dirac equation, the two chiralities represent the Lorentz-group representations $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$. What's important for now, is to notice that we end up with separate equations for the two 2-spinors, so they don't have anything to do with one another. Furthermore, if we change the frame of reference (rotation, boost), solutions of the ψ_L equation continue to be solutions of the ψ_L equation and similarly with ψ_R. It turns out that this is the defining characteristic of the term 'particle'. Think about it. A particle is described by a number of features but notably its linear 4-momentum and its angular-momentum (spin). When we change frame, those quantities change in a way governed by Poincaré transformations (Lorentz transformations plus translations). So it's natural to say that a particle is a collection of states all related by Poincaré transformations. Ultimately, this ends up as a statement that particles correspond to irreducible representations of the Poincaré group. Strangely, the very first example where we violate that definition, is the electron. But we'll get to that in a bit.

We have one more thing to talk about in order to connect with Susskind. Namely, why are the labels 'L' and 'R' used to denote the chirality? This has to do with the fact that the particles described so far, have no mass and travel at the speed of light (as explained earlier). The particles also have spin, which can be pictured as angular momentum around the direction of motion (for example, if the motion is in the z-direction, point your thumb in the z-direction and then your fingers will curl in the direction of spin – one way if your thumb points in the direction of $+z$ and the other way if it points in the direction of $-z$). We describe the spin as right or left, depending on the direction of rotation around the axis. Since the particle travels at light-speed, no change of reference frame will change the direction of spin. Moreover, that direction is determined by which representation of the Lorentz group i.e. chirality, has been used.

Another way to say all this, is that the particle chirality doesn't change with reference frame, and therefore the particle's chirality is part of the particle's definition. ψ_L and ψ_R are two different particles and they have two different directions of spin around the axis of motion.

What happens when we introduce mass? We need Lorentz invariant equations, so that limits the form of the massive Dirac equation. It generalizes the previous form as

$$\begin{pmatrix} i(\mathbf{I}_2\partial_0 + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}) & -m\mathbf{I}_2 \\ -m\mathbf{I}_2 & i(\mathbf{I}_2\partial_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}) \end{pmatrix} \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix} = \mathbf{0}. \tag{4}$$

and then, multiplying out, we get

$$\begin{pmatrix} i(\mathbf{I}_2\partial_0 + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\psi_R(x) - m\psi_L(x) \\ i(\mathbf{I}_2\partial_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\psi_L(x) - m\psi_R(x) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \tag{5}$$

We see that the mass term mixes the left and right chirality spinors, so a few things are different from the massless case:

- As time evolves, the particle-chirality can change. It is not a constant of motion.

- It is still true that a change of reference frame transforms ψ_L to ψ_L (i.e., chirality is Lorentz-invariant) and similarly with ψ_R, so technically speaking those should represent distinct particles. However, for a variety of reasons, physicists combine the two 2-spinors into a single 4-spinor, and regard these as one, rather than two, particles. If you prefer to stick to the mathematical definition of particle (as an irreducible representation), then extend the Poincaré group by adding the reflection operator (called *parity*). If we insist on irreducible representations that include the reflection operator, then we need to combine the two spinors into a single representation – in just the way that's done in the 4-spinor Dirac equation.

- If you solve the massive Dirac equation, you again end up with a Fourier transform, but instead of the 0-mass dispersion expressed by the above equation for ω_0, we end up with a dispersion relation where the energy ω_m is $\omega_m(\mathbf{p}) = \sqrt{m^2 + (p^1)^2 + (p^2)^2 + (p^3)^2}$. That expresses the motion of a particle of mass m.

We're finally ready to tackle Susskind's interpretation of the Dirac equation. Take eq. (6) and rewrite it as

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_R(x)\\ \psi_L(x)\end{pmatrix} = \begin{pmatrix}-i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} & \mathbf{0}\\ \mathbf{0} & i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\end{pmatrix}\begin{pmatrix}\psi_R(x)\\ \psi_L(x)\end{pmatrix} + \begin{pmatrix}\mathbf{0} & m\mathbf{I}_2\\ m\mathbf{I}_2 & \mathbf{0}\end{pmatrix}\begin{pmatrix}\psi_R(x)\\ \psi_L(x)\end{pmatrix}. \tag{6}$$

The second term on the right is the mass term. Suppose we solve this equation for the case when ψ is spacially independent. This situation represents a particle at rest (0 momentum). Then our equation is

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_R(x)\\ \psi_L(x)\end{pmatrix} = \begin{pmatrix}\mathbf{0} & m\mathbf{I}_2\\ m\mathbf{I}_2 & \mathbf{0}\end{pmatrix}\begin{pmatrix}\psi_R(x)\\ \psi_L(x)\end{pmatrix}, \tag{7}$$

and has the easily verified solution

$$\begin{pmatrix}\psi_R(x)\\ \psi_L(x)\end{pmatrix} = \begin{pmatrix}\alpha e^{imt} + \beta e^{-imt}\\ -\alpha e^{imt} + \beta e^{-imt}\end{pmatrix}, \tag{8}$$

where α and β are arbitrary space-independent 2-spinors.

But this is exactly what Susskind mentioned. The mass term causes an oscillation of chiralities whose frequency is proportional to the mass.

5 Condensates and the Higgs phenomenon

5.1 Overview of two pictures of the Higgs phenomenon

The way I learned about the Higgs phenomenon was mathematically straightforward and did not involve the concept of condensates. But in reading some of the literature about the history of that phenomenon, it appears the early authors were inspired by condensed matter physics, where broken symmetries have a pedigreed tradition. Moreover, there's some evidence that the early work on the Higgs phenomenon was expressed in terms of condensates.

I've made an attempt to connect the condensate picture of symmetry breaking, with the mathematics more familiar to me, and have found the literature unsatisfactory. Some authors definitely attempt to connect the two pictures, but those authors generally assume that the reader has a better understanding of condensed-matter physics than I do.

Overall, Lancaster probably has one of the best (certainly the best I've ever seen) textbook treatment of this subject. People like Susskind and others who invoke the condensate approach, have a fairly sophisticated knowledge of condensed-matter behavior including phonons, bogolons, superfluidity, phase transitions etc., and can use this knowledge to give them intuition about spontaneous symmetry breaking in quantum field theory.

I think it's worth a stab at making the connections between the condensed-matter approach and the standard textbook QFT approach. It's probably too difficult for me to do this without resorting to some math, and there's no point in doing this math without reviewing the theory in the way with which I'm most familiar. So what follows will start with the usual QFT-textbook treatment of vacuum symmetry breaking.

5.2 Ambiguity of perturbation theory

First, I think it helps to think about the big picture of what we're trying to do. Recall how we actually do QFT. We start with an expression of the full theory – for example, the path-integral expression. The object of most interest is the Lagrangian. For connecting to the Susskind talk, I'll consider a complex-scalar theory with a field that I'll call a **Higgs field** (as distinct from a *Higgs particle* which we'll come to later) whose Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) + m^2 \phi\phi^* - \lambda \phi^2 \phi^{*2}. \tag{9}$$

Although most treatments of the subject would, instead, consider a pair of scalar fields ϕ_1 and ϕ_2, for our purposes I think it suffices to rewrite the theory in terms of $\phi = |\phi| e^{i\theta} = \phi_r + i\phi_i$ where ϕ_r and ϕ_i are the 'real' and 'imaginary' components of ϕ. By the way, although you may not immediately realize this, the sign of the mass term above is opposite from what we usually show.

Since we don't know how to solve the theory exactly, we devise a perturbative approach based on the separation of the Lagrangian into a solvable (aka free) piece and the remainder. A key question is whether the resulting series behaves well enough (convergent or asymptotic etc.) to approximate the solution under certain circumstances. The series may, for some set of parameters or circumstances, stop giving even qualitatively the correct (if we could solve the theory completely) behaviors. In most cases, the series is useful as a certain parameter is changed but only up to a point. We know this kind of thing from the theory of metamorphic functions, where we often have a radius of convergence beyond which we encounter either a singularity or a branch cut. I believe (and here I'm punting because I'm not positive this is the right way to think about things) that this kind of thing manifests itself in physics via a critical point and/or a phase transition.

What do we do when we encounter a region where the series is no longer a good approximation? Again, I'm punting. But I think there are (at least) two things we can do. (a) We can start all over again by splitting the original Lagrangian into a different solvable piece with a different remainder, and then using a series of approximations to approach the solution – but in this case, with the parameters that were hitherto un-manageable in the original perturbation approach... or (b) We can attempt to re-organize the original perturbation series, perhaps by collecting terms in a mathematically justifiable way. These two approaches are really just variants of one another.

In the 'usual' textbook approach to QFT vacuum symmetry breaking, we modify the starting solvable Lagrangian as I'll show shortly. That leads to a perturbation expansion which approximates the full solution for situations encountered in our labs. I've always implicitly assumed that this new decomposition describes a different 'phase' (in the sense that liquid and solid are different phases) than the initial perturbative approach based on separating out all the quadratic terms of the original fields (see later for an explanation of what I mean). In particular, I didn't think one could start with the theory of an unbroken symmetry and derive from that, any kind of understanding of the theory of broken symmetry. Now, based on Susskind's lecture, I can see that there may be an appropriate mathematical method (akin to a reorganization of terms in the original perturbation series) that lets us organize things in such a way that we can see how an unbroken theory becomes broken – and this method is based on similar methods that were used in condensed matter physics for understanding new phases of matter such as superconductors.

5.3 A scalar theory with bad perturbative decompositions; the origin of tachyons

Write the Lagrangian as the sum of a free piece and the rest (which we call the interactive piece):

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{10}$$

where

$$\mathcal{L}_0 = (\partial_\mu \phi^*)\,(\partial^\mu \phi) + m^2 \phi\phi^*$$

and

$$\mathcal{L}_I = -\lambda \phi^2 \phi^{*2}$$

. Our perturbation expansion will be based on solving the free theory and then obtaining Green functions in the way we learned from the path integral formulation. The free theory now has propagators with **a negative mass-squared!** These can be seen to correspond to particles with the property that $E^2 - \mathbf{p}\cdot\mathbf{p} < 0$. Recall that in relativistic physics, particle momenta have the property that $\mathbf{p} = E\mathbf{v}/c$. From the inequality, this means that $\mathbf{v}\cdot\mathbf{v} > c^2$. **In other words, these particles travel faster than the speed of light! We call them TACHYONS.**

But don't get concerned. We encountered these tachyons as a result of our expansion where we chose the starting point to be $\mathcal{L}_0$ which, as we saw right away, had the wrong sign of the mass. Since tachyons violate causality, they cannot qualitatively describe the solution for the full Lagrangian. However, by drawing the potential function, we find that the potential is **not** unbounded from below and so the theory looks well-behaved. Our conclusion is that the perturbation series is ill-behaved and we need to re-organize our approximations.

5.4 The good decomposition – the usual QFT textbook treatment of symmetry breaking

But first I'll turn to the decomposition used for theories of symmetry breaking. I don't know how to draw pictures in Latex and I'm too lazy right now to insert pdf files created with powerpoint. So I'll ask you to draw the function $V(x) = -m^2x^2 + \lambda x^4$ where $m^2 > 0$ and $\lambda > 0$. This function rises steeply as x gets very positive or very negative. There is also a local maximum at $x = 0$ and the function descends into a trough on both the positive and negative sides of the x-axis. We call this *the Mexican hat potential.* The idea for the perturbation expansion, is to pick a point in the trough and expand around it. Notice that although I have written the potential as a function of

x, this is highly misleading. You are supposed to think of x as representing $\phi(x_0, x_1, x_2, x_3)$. Pick one of the minima of $V(x)$, for example $x = C = \frac{m}{\sqrt{2\lambda}}$ (you can find this minimum by taking the derivative and setting it to zero). Let's revert back to the notation with fields, and expand around $\phi(x) = C$. One way to do this is to define a new field $\phi'(x)$ by $\phi(x) = \phi'(x) + C$. Then, if we expand the Lagrangian in terms of $\phi'(x)$ we get

$$\mathcal{L} = (\partial_\mu \phi')^* (\partial^\mu \phi') - \frac{m^2}{2} \left(\phi' + \phi'^*\right)^2 - \sqrt{2\lambda}\phi'\phi'^* \left(\phi' + \phi'^*\right) - \lambda \phi'^2 \phi'^{*2} + \frac{m^4}{4\lambda}$$

We can again separate the entire quadratic part of the Lagrangian as the starting point for the perturbative expansion. This time, the mass term is positive and the rest of the potential – except for the overall constant which plays no physical role – is positive, and the perturbation expansion is a good approximation. However, one of the symmetries of the original Lagrangian is now hidden. This new Lagrangian involves the real part of the field ϕ' whereas the original Lagrangian only involved the magnitude of ϕ. Things get even more interesting when coupling the ϕ field to other fields such as electrons. Terms like $\bar{\psi}\phi\psi$ get expanded to become, for example $\bar{\psi}\phi'\psi + C\bar{\psi}\psi$. The second term is a mass term. So even if the original fermion Lagrangian had no mass term, the scalar coupling leads after shifting, to a mass term for the fermion. Hence the origin of masses.

5.5 Naive perturbation theory, particles, charges, condensates, Ziggs and Higgs

Another Lagrangian decomposition we can use starts with a free massless scalar theory.

$$\mathcal{L}_0 = (\partial_\mu \phi^*) (\partial^\mu \phi)$$

and

$$\mathcal{L}_I = m^2 \phi\phi^* - \lambda \phi^2 \phi^{*2}.$$

Now the free theory we start with has massless particles. Those are much better-behaved than tachyons, so they offer some hope of a better set of approximations. In the usual way, the free field $\hat{\phi}(x)$ can be expanded as

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2|\mathbf{p}|}} \left(a(p)e^{-ipx} + b^\dagger(p)e^{ipx}\right). \tag{11}$$

When the operator $a^\dagger$ is applied to the vacuum, it gives a one-particle state. We say that $a^\dagger$ creates 'particles'. Traditionally, we say that $b^\dagger$ creates 'antiparticles'. However, for our purposes we should just think of them as two different kinds of 0-mass particles.

Some of what follows will parallel a treatment of Lancaster's, primarily found in chapter 42. However, Lancaster is dealing here with condensed matter physics,so I'll try to re-cast things in terms of quantum fields.

Define the coherent state $|\alpha,\theta\rangle$ by

$$|\alpha,\theta\rangle = Ne^{\int \frac{d^3p}{(2\pi)^3}\alpha(p)\left(a^\dagger(p)\cos\theta(p)+ib^\dagger(p)\sin\theta(p)\right)}|\Omega\rangle$$

where N is a normalization factor such that $\langle\alpha|\alpha\rangle = 1$[4]and $\alpha(p)$ is real and positive. I'm going to refer to this state as a condensate, although I don't know whether this is an accurate term, nor am I sure it's what Susskind had in mind. But for some particular value of the function $\alpha(p)$, I think it might be the condensate. The key feature of this state, is that it is an eigenstate of the annihilation operator: $a(p)|\alpha,\theta\rangle = \alpha(p)\cos\theta(p)|\alpha,\theta\rangle$. Similarly, $\langle\alpha,\theta|b^\dagger(p) = i\alpha(p)\sin\theta(p)\langle\alpha,\sin\theta|$. One interesting interpretation of all this, is that because the annihilation operator doesn't change this coherent state, it is effectively removing a particle from the state without making a difference. That's a point that Susskind makes about the condensate.

Now we're ready to evaluate $\langle\alpha,\theta|\hat{\phi}(x)|\alpha,\theta\rangle$. From the previously given expansion of $\hat{\phi}(x)$ and the eigenstate equations for the annihilation and creation operators, we get

$$\langle\alpha,\theta|\hat{\phi}(x)|\alpha,\theta\rangle = \int \frac{d^3p}{(2\pi)^3\sqrt{2}|\mathbf{p}|}\alpha(p)\left(\cos\theta(p)e^{-ipx} + i\sin\theta(p)e^{ipx}\right).$$

We call the RHS of this equation $\phi(x)$ (without the hat).

Return to our Lagrangian. The energy operator is

$$\hat{E} =: \int d^3x \left(\partial_0\hat{\phi}^*\partial_0\hat{\phi} + \nabla\hat{\phi}^*\cdot\nabla\hat{\phi} - m^2\hat{\phi}^*\hat{\phi} + \lambda\hat{\phi}^{*2}\hat{\phi}^2\right): \qquad (12)$$

The colon symbols which bracket the RHS, denote normal-ordering. That's something I haven't really discussed previously and which requires some careful thinking. Briefly, when promoting fields to operators, there is an ambiguity of how operator products should be ordered in the Lagrangian. If the ordering is changed, then commutation relations will imply some new terms. This is dealt with, in modern QFT, by normal-ordering. This concept is defined by decomposing the fields into their constituent annihilation and creation operators, then multiplying out the fields into sums of products of annihilation and creation operators, and then finally re-ordering operators within each product so that the annihilation operators appear on the left.

[4]I don't actually know whether this state is normalizable, but if it isn't, I'm sure we can fiddle around and come up with something acceptable.

When we take the expectation value of the energy operator in a coherent state, we obtain

$$\langle\alpha,\theta|\hat{E}|\alpha,\theta\rangle = \int d^3x \left(\partial_0\phi^*\partial_0\phi + \nabla\phi^*\cdot\nabla\phi - m^2\phi^*\phi + \lambda\phi^{*2}\phi^2\right). \qquad (13)$$

Alternatively, we could look at the energy-density operator and when we compute its expectation value in the state $|\alpha,\theta\rangle$, we'd obtain the integrand of the above expression.

We see that with this procedure, field operators get converted into field values by acting on the appropriate coherent states. The theory of coherent states was developed by Glauber for electromagnetism, and provides a straightforward connection between the classical electromagnetic fields, and expectation values of field operators. Furthermore, these coherent states can be pictured as many-body systems and that allows us to apply some intuitions that come from many-body physics, particularly as they cool down to condensates.

Because we are now dealing with complex-valued functions (rather than operator-valued functions), we can make sense of terms such as 'vibrations of the field'. For each $|\alpha,\theta\rangle$ (recall that $\phi(x) = \langle\alpha,\theta|\hat{\phi}(x)|\alpha,\theta\rangle$), $\phi(x)$ can be regarded as a sum of coefficients times wave-phases for each 3-momentum (and, noteworthy, the phases obey 0-mass momentum-dispersion relations).

Unfortunately, that picture leads to statements like 'particles are field vibrations'. Maybe that's a valid statement, but to my taste it's a bit of a stretch. The field is an operator. When it acts on the naive vacuum (by which I mean the state which is the lowest-energy state of the free theory) it creates a superposition of particle states each multiplied by an oscillating phase. In my opinion, that's not the same thing at all as saying that the particle is, in some fashion, a perturbation of the field.

Be that as it may, let's return to the coherent-state picture. We know from our experience with the construction of particle states out of free fields, and also from our experience in condensed-matter physics, that the most important state is the lowest-energy state – and that from there, physics is developed from the small perturbations to that state. So let's find that state.

The lowest-energy (normalized) state $|0\rangle$, or vacuum, will have the property that

$$\langle 0|\hat{E}|0\rangle$$

is the minimum among all states. However, we'll see that this minimum can be achieved by more than one state, so we will eventually change our notation for the vacuum in this theory.

Assume (I think this may be true but don't know for sure) that the coherent states $|\alpha,\theta\rangle$ are 'complete'. That is, all states of the Hilbert space can be generated by them. Then it will suffice to minimize $\langle \alpha, \theta | \hat{E} | \alpha, \theta \rangle$ over all coherent states $|\alpha, \theta\rangle$. Namely, find the minimum of

$$\langle \alpha, \theta | \hat{E} | \alpha, \theta \rangle = \int d^3x \left(\partial_0 \phi^* \partial_0 \phi + \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi + \lambda \phi^{*2} \phi^2 \right) \tag{14}$$

Here, $\phi(x)$ is simply a complex-valued function. Since the derivative terms are positive, it's easy to see that the minimum will be achieved when $\phi(x)$ is a constant. In this situation, the integral will be infinite. Obviously, that situation will need to be treated with care and once again, I don't know what happens when rigor is applied. I'll adopt the usual tactic of saying that we only integrate over a finite large volume $\mathcal{V}$. Since the functions are constant, we then get

$$\langle \alpha | \hat{E} | \alpha \rangle = \mathcal{V} \left(-m^2 \phi^* \phi + \lambda \phi^{*2} \phi^2 \right).$$

Rewrite this as

$$\langle \alpha | \hat{E} | \alpha \rangle = \mathcal{V} \left(-m^2 |\phi|^2 + \lambda |\phi|^4 \right).$$

It's easy to minimize this function of (the constant) $|\phi|$. Take the derivative $\frac{d}{d|\phi|}$ and set the result to 0. Noting that $|\phi| \geq 0$, we obtain 2 solutions, only one of which is a minimum. Namely,

$$|\phi_{\min}| = \frac{m}{\sqrt{2\lambda}}.$$

You should recognize this value as the constant C from last section.

This quantity is sometimes known as the vacuum expectation value, or VEV, and plays an important role in the weak and electromagnetic interactions. It has been measured to have a value of 246 GeV.

Importantly, we see that the vacuum is degenerate. Let

$$\phi(x) = \frac{m}{\sqrt{2\lambda}} e^{i\theta}.$$

Then, for any constant phase θ and constant real value α, the corresponding state $|\alpha, \theta\rangle$ (recall that $\phi(x)$ is related to α and θ via a Fourier transform) is a vacuum. If we recall the Mexican hat potential, think of the height of the hat as $|\phi|$ and the angle around the hat as θ. The distance from the center of the hat to the trough is the VEV.

Let's recap. We decomposed the Lagrangian into a piece which describes a massless scalar particle, and a piece which describes the interaction term (and includes the mass term). The interaction term, if expressed with complex-valued fields, looks like a Mexican hat. We then create coherent states out of the massless scalar particle states, and search for coherent states that minimize the system energy. These states can be expressed as functions of space-time which are Fourier transforms over definite-momentum states. The minimum-energy coherent states – also called condensates – have the property that they are constant in space and time. The condensates differ from one another by a phase θ. Any one of them can be regarded as a vacuum state. One other observation is worth making. For ϕ to be a constant, the only term present in the Fourier expansion is the 0-momentum term. So going back to the original definition of the coherent state, we see that

$$|\alpha, \theta\rangle_{\min} = N e^{\alpha\left(a^\dagger(0)\cos\theta + i b^\dagger(0)\sin\theta\right)}|\Omega\rangle. \tag{15}$$

In the original coherent-state definition, the above would require that $\alpha(p) = \alpha\delta(p)$. (As always, we'd need to eventually make sense of this by using some mathematical legerdemain.) The vacuum state would then be interpreted as a weighted sum over states consisting of n scalar particles with 0 momentum. The weighting factor is proportional to the VEV. We will need to pick a vacuum, so **from now on, pick $\theta = 0$, in which case $|0\rangle \equiv |\alpha, 0\rangle_{\mathbf{min}} =$** $Ne^{\alpha a^\dagger(0)}|\Omega\rangle$. The free 0-momentum particle which 'condenses' in that vacuum state is $a^\dagger|\Omega\rangle$ which Susskind calls the ***Ziggs*** particle.

A popular analogue in condensed matter physics is obtained by considering a substance of magnetic spins which, at high temperature, are lined up randomly. The theory is spherically symmetric and therefore no direction is preferred to any other. However, as the substance is cooled, the spins tend to line up with one another. In that particular example, some minor external perturbation is undoubtedly responsible for the initial impulses that pick one direction over another. But the cooler the substance becomes, the more the spins line up with one another until at 0 temperature, they are lined up in one direction (in 2D, we call it the phase angle θ). If we started all over again from a hot substance, the cooled substance would look the same, except the spins would line up along a different phase angle.

Let's turn back to Susskind. In the condensed matter spin example of the previous paragraph, we saw that a high temperature, the average spin was 0 by spherical symmetry. But when cooled, the average spin acquires a non-zero value. All spins line up in some direction. This clearly breaks the spherical symmetry so we describe this by saying **'the symmetry is broken by the vacuum'**.

5.6 Fermions and Z particles in the condensate

Now that we have some kind of condensate description of the vacuum, Susskind talks about the behavior of an electron, for example, in that condensate.

Recall that Susskind says left-handed electrons carry a certain Zilch and right-handed electrons carry a different Zilch. Susskind then tells us that there is an interaction term in which a right-handed electron can emit a (0-momentum) Ziggs particle and become a left-handed electron. In the presence of the vacuum condensate, the emission of a Ziggs just creates a Ziggs in the condensate, and as previously discussed, this Ziggs gets absorbed into the condensate without changing the condensate. Similarly, a Ziggs can be absorbed by a left-handed electron, to become a right-handed electron. The situation is very much like the one that Susskind discusses at the very beginning of his lecture, of a dipole traveling through an electric field (and as pointed out earlier, an electric field is a coherent photon state), which can be thought of as a dipole interacting with photons that are freely (i.e., without impact) absorbed and emitted from/to the field.

So now, we have an electron traveling through the condensate, and changing from left to right and back again, at a rate determined by the strength of the interaction. This oscillation between left and right was earlier – in the discussion of the Dirac equation – shown to give mass to the electron.

What remains of the Higgs field particles? When we consider coherent states of non-zero momentum, there are oscillations of terms involving $\alpha(p)$ and $\theta(p)$. When we evaluate the energy operator of the fully interacting theory, we'll see that there are energy terms coming from motion in the radial direction (magnitude of $\alpha(p)$) and terms coming from motion in the angular direction $\theta(p)$. It turns out that the angular terms appear to have 0 mass and the radial terms have a mass proportional to m in the original Lagrangian. Those latter terms are then interpreted as particles in the presence of the condensate, and those particles are called **Higgs particles**. What about the 0-mass modes? These modes are called Goldstone particles, but they don't appear in our world. Why not?

There's a subtle trick (I'm not convinced Susskind does this justice). It turns out that there are Z-bosons in the theory, with which the Higgs field can interact. Through a change of field variables, we can come up with a Lagrangian that describes a Z'-boson, and only the Higgs modes which have non-zero mass. This re-jiggering of the fields sometimes goes by the expression 'the Z mesons swallow the Goldstone bosons'. But the interaction between Z's and Higgs' also gives rise to the sort of thing which gave mass to the fermions. Ziggs particles can be absorbed and emitted by the Z's, and like with the fermions, these essentially disappear into or out of the condensate. The net effect, which can be combined with the Goldstone sword-swallowing act, is to give the Z' bosons a mass.

Having said all this, I'm not positive that this condensate-based picture is entirely correct. I'm much more comfortable dealing directly with the

Lagrangian and shifts of variables, and as you can see from the last few paragraphs, I tend to revert to that way of thinking about things even in the context of condensates. But I believe I've more or less bridged the two pictures of spontaneous symmetry breaking.