A comment about notes on the spectrum of the hydrogen atom

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In the section immediately preceding “Important Algebraic Result 1”, I remark that a scalar operator $\mathbf{O}$ is defined to be an operator that has the property $\mathbf{ROR}^{-1} = \mathbf{O}$. This remark led to several questions during our discussion, and I’ve concluded that I was too sloppy and too glib. So here’s a more complete version.

We have a Hilbert space of states ψ and a ‘representation’ of rotations on that space, by which I mean linear transformations $\psi \to \psi'$ which can be described by a rotation-dependent operator $\mathbf{D}(\mathbf{R})$ – namely, $\psi' = \mathbf{D}(\mathbf{R})\psi$. (If our space were finite-dimensional (which it is not), then we would describe the vectors as an n-tuplet and the operators would be matrices.)

We think of $\mathbf{D}(\mathbf{R})$ as something which performs rotations (this interpretation is called ‘active’) – or alternatively, you can think of this as resulting from a transformation of coordinates (this interpretation is called ’passive’ and requires that we replace the rotations by their inverses). When we describe abstract Hilbert states using wavefunctions $\psi(\mathbf{x})$, the operator $\mathbf{D}(\mathbf{R})$ takes the state described as $\psi(\mathbf{x})$ to the state $\psi'(\mathbf{x}) = \psi(\mathbf{R}^{-1}\mathbf{x})$. (*Hopefully I haven’t screwed up my inverses here, but if you think I have, then you might be right!*)

Now consider a rotationally invariant (a.k.a. ‘scalar’) operator $\mathbf{O}$. By definition of ‘rotationally invariant’, this operator should give the same measurement regardless of how you rotate the coordinate system. Since measurements in QM are given by inner products denoted by a pair of parentheses (...), our statement of invariance becomes $(\mathbf{D}(\mathbf{R})\psi, \mathbf{OD}(\mathbf{R})\psi) = (\psi, \mathbf{O}\psi)$. By properties of the inner product, this is the same as writing $\left(\psi, \mathbf{D}^\dagger(\mathbf{R})\mathbf{OD}(\mathbf{R})\psi\right) = (\psi, \mathbf{O}\psi)$ where $\mathbf{D}^\dagger(\mathbf{R})$ is the adjoint of $\mathbf{D}(\mathbf{R})$. Since this is true for every state ψ, we then conclude that $\mathbf{D}^\dagger(\mathbf{R})\mathbf{OD}(\mathbf{R}) = \mathbf{O}$. Furthermore, since the identity operator $\mathbf{I}$ is clearly a scalar (rotationally invariant) we also have

$\mathbf{D}^\dagger(\mathbf{R})\mathbf{D}(\mathbf{R}) = \mathbf{I}$, implying that $\mathbf{D}^\dagger = \mathbf{D}^{-1}$. Thus $\mathbf{D}(\mathbf{R})$ is a unitary operator.

Anyway, I've finally arrived at the expression that should have been written instead of $\mathbf{ROR}^{-1} = \mathbf{O}$. Namely, $\mathbf{D}^{-1}(\mathbf{R})\mathbf{O}\mathbf{D}(\mathbf{R}) = \mathbf{O}$. (Or I could have used the adjoint instead of the inverse.) In this more accurate notation, the angular momentum operators L_i are to be regarded as infinitesimal generators of the representations $\mathbf{D}(\mathbf{R})$, i.e., ***representations*** of the Lie Algebra of rotations, rather than elements of the Lie Algebra.