# Ruminations on relativity

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## 1 Why geometry?

### 1.1 Quick and dirty

Start by pretending that the earth's surface is a perfectly smooth sphere. You and a friend start off at the equator 100 miles apart heading northward perpendicular to the equator on parallel routes. Each of you walks in a straight line (”as straight as possible on the surface of a sphere”). Eventually, you will discover that you are only 50 miles apart (and ultimately you both end up on top of each other at the north pole). Lesson: “objects moving in parallel can change their distances from one another (on the earth).”

So far, so good. What we've established is that non-flat geometry can change our concepts of how straight lines behave. Now let's talk about gravity. The key example here is the falling elevator. But just to bring things into the space age, consider instead that you have the same perfectly spherical earth – with uniform volume – and two space labs a distance 100 miles from the earth and 10 miles apart from one another. Both are traveling towards the earth's center at the same speed as one another. In each lab, physicists are testing Newton's first law. They take a ball and suspend it in their lab. It doesn't move. Newton's first law is confirmed. But now they look our the windows of their labs and look at what's going on in their neighbor's lab. Let's pretend they can only look horizontally so they don't see the earth (and in fact don't even know the earth is there). What they'll notice, is that the two balls (each in their respective lab) are drifting towards one another in apparent violation of Newton's law. You might object that this example is too contrived since clearly the two labs are converging on one another. So instead, imagine the two labs as collinear and falling towards the earth. One is 100 miles from the earth and the other is 1000 miles from the earth. Again, in each lab, the balls stay put. But if the two experimenters are able to see each other's balls with a supertelescope, they'll notice that the balls are accelerating towards one another.

Bingo! This is the same phenomenon as we saw with the parallel lines on the sphere. Instead of "travel in as straight a line as possible northward" (for the sphere) we have "keep a ball suspended without an apparent acting force". In both cases, when comparing notes to a traveler or experimenter some distance away, we discover that there is relative displacement or motion.

Leap of faith! The case of gravity exhibits characteristics of the case of the sphere, hence gravity is simply a manifestation of geometry.

## 1.2 Details, details

In point of fact, the two above examples share slightly more mathematical formalism than indicated. In order to be precise about the sphere, we need to say what we mean by "a straight as possible". As usual, the mathematics provides an abstraction of intuitive concepts. Since limits are required, the formalism tends to make demands on how various derivatives behave. In any case, one common and relevant intuition, is that the traveler on the sphere has two large flat pieces of paper with straight lines on them, and then lays out the papers on the ground (obviously bending it just a tiny bit to fit the earth's contour). The two pieces overlap one another a little bit, and the traveler makes sure the lines overlap one another 'as much as possible'. Then after traveling along the two pieces of paper following the line, the first paper is removed and placed in front of the other and the process is repeated. This intuitive approach becomes the key to the mathematical definition of a manifold and ultimately a geometry. The pieces of paper are thought of as charts or maps (think of Mercator projection) and the conditions determining the calculability of quantities on the manifold, are conditions having to do with behavior of overlapping charts.

Just like the travelers on the sphere carry their own charts (the pieces of paper with straight lines), the experimenters in their labs carry their own charts – in this case equipment capable of determining that a ball isn't moving (as much as possible). **HOWEVER**, we haven't established that a chart-like behavior necessarily implies a geometric surface.

Mathematically, we need to impose certain conditions on the behavior of experiments in spacetime and how those experiments relate to one another when comparing them at a finite distance (and that requires a mathematically precise notion of distance). IF those conditions hold, then spacetime can be described as a geometry in just the same way that the earth's surface is described as a geometry.

As it turns out, the mathematical conditions leading to geometry, seem pretty reasonable. But it could be important to keep in mind just what assumptions lead to the geometrical interpretation. Steven Weinberg, in his book on gravitation, makes a concerted effort to show what those assumptions are and what physical things they correspond to.

There's nothing new about this kind of thing in physics. Classical mechanics relies heavily on assumptions about differentiability. Few physicists spend much time questioning those assumptions. On occasion, when it becomes necessary to introduce discontinuous functions (e.g. phase transitions), phsyicists and mathematicians have come up with formulations that are superimposed on more familiar differentiable structures.

In the same way, we find that most practicing physicists at this point, accept the geometrical picture of spacetime as the mathematical foundation for gravity. I think that if there were failures of the geometrical picture, they would probably be treated as some kind of limiting case of geometries. Maybe a good example to consider is the Ptolemaic theory of the universe where stellar and planetary motions were regarded as circles superposed on circles. Although the ancients didn't know about Fourier series, we now know that the underlying mathematics has to do with periodic motion and that one can approximate periodic orbits by an infinite series of circular motions. If, eventually, too many complexities are required for a proper interpretation of gravitational geometry, then that could lead to a different way of thinking about things, where geometry is just an approximation.

In some sense, we may already be seeing some motion towards a 'derivation of geometry'. String theory, which is based on a 2D geometry (and is therefore not obviously related to spacetime) leads, on the basis of some very simple assumptions to Einstein's (4D geometric) equations. But of course, that still leaves the question of "how do you derive string theory?" (i.e., what is the motivation, the evidence and the set of axioms regarded as 'reasonable'?).

# 2 Tensors

Once we are convinced that the right arena for gravity is geometry, then our fundamental rule is this: at every point in spacetime, one can find a chart (i.e. coordinate system) where the laws of physics are precisely the ones which we get when there are no gravitational forces. For example, Maxwell's equations. But the mathematicians have taught us a way to write those equations involving operators and quantities that are 'native' to the geometry (for example, we know that a great circle on a sphere is a straight line on a local chart – so the 'native' quantity corresponding to a straight line, is the great circle). Many of the laws of physics involve indices (having to do with Lorentz transformations) and the 'native' quantities also involve indices. But those native quantities are tensors and, unlike the case of flat space, upper indices and lower indices may be related to one another via a complex metric tensor. Even more important is the 'native' interpretation of differentiation. (Think about how you'd want to define differentiation along a great circle.) This leads to the idea of covariant derivatives and the connection symbols (Christoffel or Levi-Civita) – the connection symbols being a necessary requirement for 'connecting' maps at two different base points.

Just as, in relativistic physics, Lorentz invariant equations require the same tensor structure on both sides of the equation, so in general-relativistic physics, the geometry requires the same tensor structure on both sides.

Now for the 'derivation' of Einstein's equations. Einstein knew that the source of gravity – i.e, the source of the geometry – was mass. He also knew that mass and energy were the same thing and that the correct Lorentz tensor (if you want to construct a Lorentz-invariant theory) was the energy-momentum vector. Another aspect of a relativistic theory, is that it has to sensibly handle the notion of action at a distance, for which it makes much more sense to speak of something like a field – to wit, the energy-momentum 'density' – which is treated relativistically as the 'energy-momentum tensor'.

So, on the right side of the equation you have the energy-momentum (2-index) tensor and on the left side, you have some Lorentz-covariant tensors containing combinations of the metric tensor and their derivatives. Now we can impose some restrictions on what kinds of terms are valid. For example, we might insist that the left-hand side has the same dimensions as the RHS but WITHOUT requiring the introduction of any new dimension-ful constants (beyond G, c etc.). To see a set of reasonable assumptions, look at Weinberg section 7.1. Under those assumptions, we end up with the LHS consisting of combinations of the Ricci tensor, Ricci scalar and metric tensor. Then with a few other assumptions – including the necessity of matching up with classical gravity for slow objects – Einstein's equation can be derived.

# 3 Heisenberg versus Schrodinger states

Strangely, while thinking of general relativity, I bumped into an issue germane to field theory (and to my current readings on quantum field theory in curved spacetime). Remember that in old-fashioned non-relativistic quantum mechanics, there were two important equivalent formulations of quantum mechanics. In one formulation, known as the 'Schrodinger picture', states evolve in time, and operators are regarded as time-independent objects (if you measure 'position', the act of measurement shouldn't depend on when the measurement is done). In the other formulation, the states are time-independent (the 'state of a system' should have the same meaning no matter when you look at it) but the operators evolve in time. The two pictures are related to one another by unitary operators and the reason the two pictures are equivalent, is that the only physically meaningful quantities are ones where you perform a measurement on a state – so as long as one or the other evolves with time, you get the same answer (but don't double-count!)

I've tended to think that the same kind of pictures could be used in relativistic quantum mechanics and formally, since the two pictures are related via unitary transformations, it's true that one could derive one picture from the other.

However, just like with general relativity, it's really important to think about what various terms 'mean'. A relativistic formulation of quantum mechanics cannot (easily) make 'time' special, compared to 'space'. In any straightforward generalization of non-relativistic quantum mechanics, the distinction between time and space poses a problem. For example, in n-r q.m., position is treated as something observable whereas time is a parameter.

What field theory does, is to change the problem statement to one where the operators are both time and space dependent and where time and space are both treated as parameters. This is a non-trivial generalization. But in particular, it is a generalization of the Heisenberg picture – not of the Schrodinger picture.

So in field theory, states are both time and space independent. What does that mean? It does **NOT** mean that the state is regarded as something occurring at a single point in time and space. In particular, one should not picture the state as containing information about the system for all points of spacetime in the universe.

But what does it mean to say the system evolves? Unfortunately, our ideas of evolution tend to track more closely with the Schrodinger picture, in which the state of the system changes with time. For example, when we speak of a scattering experiment, we consider a particular state in the far past, then watch it evolve into the far future. But that's not a Lorentz invariant concept. In a different frame of reference, the original 'state in the far past' is extended across the universe but at different times – since what was simultaneously at time T in one reference frame, isn't simultaneous in the other reference frame. In fact, this very issue is what leads to the behavior known as entanglement (or more precisely, the behavior known as entanglement is potentially surprising when attempting to reconcile interpretations with relativity).

As it turns out, the mathematical formulation is consistent (or appears to be, although the famous black hole information paradox might cast doubt on consistency) so allows us to punt on the various interpretations. For example, the scattering problem is posed formally as a problem where particles are created in the far past (in whatever reference frame is chosen) by the operation of field operators on THE vacuum which happens to be frame-independent.

What wasn't really noticed or commented upon until the 1960's, is that the very concept of particles in field theory, is only consistent amongst inertial frames of reference (i.e. frames related to a rest frame via a Lorentz transformation). In non-gravity physics, there is no reason to ever consider anything other than inertial frames, so there was no ambiguity about the meaning of particles. But once we admit gravity into the picture, then we allow generalization of all concepts to non-inertial frames. So, for example, if you are doing an experiment in an accelerating rocket, you could pick your frame of reference to be the rocket. Since it's accelerating, you don't observe Newton's first law, but since you are a master mathematician, you can compensate by figuring out the effect of the acceleration of your rocket. One of the peculiar effects of acceleration is that you might count the number of particles you see differently than if you passed by a rocket at rest where the experimenter just happened to see what particles were in your lab. Like I said, you could do all the appropriate mathematics and figure out the relationship between the number of particles you see and those seen in a rest frame. But what if instead of acceleration, you were experiencing a gravitational field? You'd get the same results as acceleration. But there'd be no reason to relate it back to an inertial frame since, after all, the 'real world' is the one where there is gravity.

What you're left with, is the fact that particle-number is frame-dependent. This leads to the very strange fact that particle-number is also dependent on the gravitational field. But what is really going on is simply that you need to pick a reference frame. The full generality of general relativity doesn't actually tell you what reference frame to pick. That's something 'extra' and has to do with convenience.