The spectrum of the hydrogen atom – an algebraic approach

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1 Overview

The object of this discussion is to explain how spectral degeneracy and spectral splitting are related to symmetry and Lie Algebras. This will later motivate the discover of 'flavor SU(3)' as a symmetry of nature that could explain the mass multiplets of hadrons.

Personal note: The core of this presentation was inspired by my T.A. Erick Weinberg, teaching a weekly section in 1972, to augment our graduate Quantum Mechanics course. The lecture was hugely exciting to me because it reminded me of what I'd entirely forgotten about becoming a physicist. My interest in physics was essentially created when, in 1960 at age 10, I read a book by Asimov called "Inside the Atom". Asimov introduced to me the great mystery of what became known as 'the particle zoo'. Over the years, I forgot that this was the root of my decision to become a physicist, and it all came back to me after Eric's lecture.

- The eigenvalue equation for a Coulomb potential, first using Cartesian coordinates, then spherical coordinates.
- Rotational symmetry, separation of variables, spherical harmonics, spectrum degeneracy and irreducible representations illustrated with spherical harmonics.
- **Angular-momentum multiplicity and rotation algebra.**

2 The time-independent Schrodinger eigenvalue equation for the Coulomb potential

(This material is standard in QM books.) Recall the general time-independent Schrodinger equation (sometimes we call it the eigenvalue equation)

$$\left[\frac{\mathbf{p}^2}{2m} + V(\mathbf{r})\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{1}$$

The QM prescription substitutes for $\mathbf{p}$, the operator $-i\boldsymbol{\nabla}$. The Coulomb potential is $V(\mathbf{r}) = -\frac{e^2}{r}$, so the Schrodinger equation becomes

$$\left[-\frac{\boldsymbol{\nabla}^2}{2m} - \frac{e^2}{r}\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{2}$$

Since the potential is spherically symmetric, and the kinetic term is also spherically symmetric, it's convenient to use spherical coordinates.

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned} \tag{3}$$

Then the Schrodinger equation becomes

$$\left[-\frac{1}{2mr}\frac{d^2}{dr^2}r + \frac{\mathbf{L}^2}{2mr^2} - \frac{e^2}{r}\right]\psi(r,\theta,\phi) = E\psi(r,\theta,\phi). \tag{4}$$

where

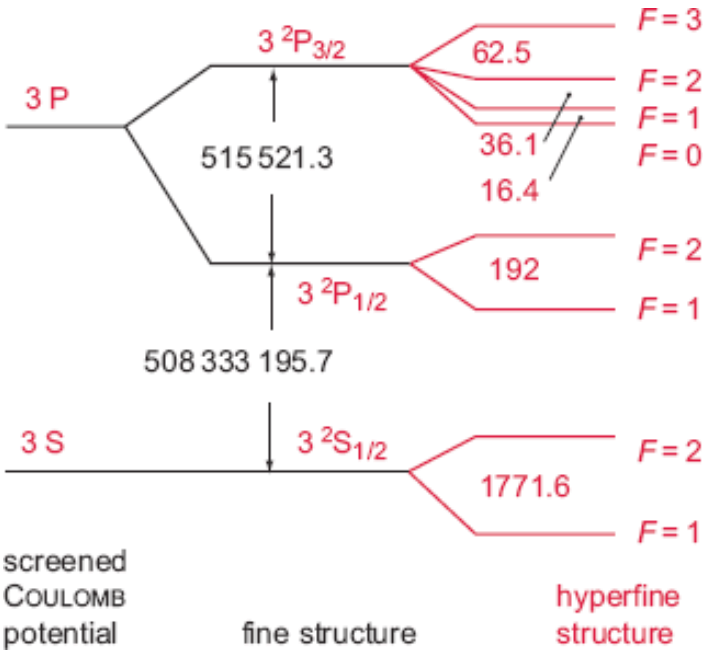
$$\mathbf{L}^2 = -\frac{1}{\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}\right]. \tag{5}$$

NOTICE the introduction of the notation $\mathbf{L}^2$. This notation will be explained later.

3 Rotational symmetry, spherical harmonics etc.

Begin by noticing that the original Schrodinger equation is spherically symmetric since the operator depends only on the radial distance r and the scalar $\mathbf{p}^2$. What this means, is that if $\psi(\mathbf{r})$ is a solution, then so is the function $\psi'(\mathbf{r}) = \psi(\mathbf{Rr})$ where $\mathbf{Rr}$ is the result of rotating the vector $\mathbf{r}$ by the rotation matrix $\mathbf{R}$. **In particular, if there is an eigenvalue**

E corresponding to $\psi(\mathbf{r})$, then the same eigenvalue corresponds to the rotated function $\psi'(\mathbf{r})$. Rotations cause *energy-degeneracy*.



Ultimately, this symmetry implies that the Schrodinger equation can be solved by the separation of variables. To see this, note that the Schrodinger operator is a sum of terms that depend only on the distance r and on the angles. First solve

$$\mathbf{L}^2 f_\alpha(\theta,\phi) = \alpha f_\alpha(\theta,\phi) \tag{6}$$

and then solve

$$\left[-\frac{1}{2mr}\frac{d^2}{dr^2}r + \frac{\alpha}{2mr^2} - \frac{e^2}{r}\right]\chi_\alpha(r) = E_\alpha \chi(r). \tag{7}$$

The net result is

$$\left[-\frac{1}{2mr}\frac{d^2}{dr^2}r + \frac{\mathbf{L}^2}{2mr^2} - \frac{e^2}{r}\right]\chi_\alpha(r)f_\alpha(\theta,\phi)(r) = E_\alpha\chi_\alpha(r)f_\alpha(\theta,\phi), \quad (8)$$

which is the same as the Schrodinger equation eq. (4) with eigenvalue E_α and eigenfunction $\psi(r,\theta,\phi) = \chi_\alpha(r)f_\alpha(\theta,\phi)$.

We begin by dealing with the angular part, eq. 6. It turns out that this equation has eigenvalues $l(l+1)$ where l is an integer ≥ 0. For every value of l, there are multiple orthogonal (with respect to an angular metric) solutions called spherical harmonics, and these are indexed by m.

$$\mathbf{L}^2 Y_l^m(\theta,\phi) = l(l+1)Y_l^m(\theta,\phi). \quad (9)$$

Since m is an index denoting multiple solutions for a single eigenvalue of $\mathbf{L}^2$, we see that this causes an energy degeneracy of $E_{l(l+1)}$. A general solution of the above eigenvalue equation, for a given eigenvalue $l(l+1)$ is a linear combination of Y_l^m. Here are some examples of Y_l^m (with normalization factors that won't matter for us):

$$\begin{aligned} Y_0^0 &= \sqrt{\frac{1}{4\pi}} \\ Y_1^{-1} &= \sqrt{\frac{3}{8\pi}}e^{-i\phi}\sin\theta \\ Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_1^1 &= -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta. \end{aligned} \quad (10)$$

Let's see what happens if you rotate the arguments of $Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$ by 90 degrees around the x axis. This rotation transforms the vector whose polar coordinates are (θ,ϕ) into the vector represented by (θ',ϕ') such that $\cos\theta' = -\sin\theta\cos\phi$.

So

$$\begin{aligned} Y_0^1(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \to -\sqrt{\frac{3}{4\pi}}\sin\theta\cos\phi &= \frac{1}{\sqrt{2}}\left[\left(-\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta\right) - \left(\sqrt{\frac{3}{8\pi}}e^{-i\phi}\sin\theta\right)\right] \\ &= \frac{1}{\sqrt{2}}\left[Y_1^1(\theta,\phi) - Y_1^{-1}(\theta,\phi)\right]. \end{aligned} \quad (11)$$

Since both the LHS and RHS are linear combinations of Y_1^m's, they are both eigenfunctions of $\mathbf{L}^2$ with eigenvalue $l(l+1) = 2$. **This turns out to be a special case of what was noted earlier – that the rotational symmetry of the LHS of the eigenvalue equation, implies eigensolutions of that equation will transform into one another.**

Although I haven't proven this, it's plausible that the Y_1^m's form the basis of a vector space whose elements are all linear combinations of Y_1^m, and what we see is that the rotations cause vectors of this space to be transformed to new vectors of this space. The transformations are linear so can be described as matrices. We call these *linear representations* of the rotation group and we call the vector space an *irreducible* representation space (of the rotation group) of dimension 3 (since there are, in this case, 3 basis vectors – one for each value of m). It's called 'irreducible' because no subspace can be found which transforms under rotations only into itself (that's a fancy way of saying that no matter which combination you start with, you can rotate to any other combination). It turns out that for each value of l, the spherical harmonics Y_l^m generate an irreducible representation space for the rotations.

In summary:

- You can solve the eigenvalue equation $\mathbf{L}^2 f(\theta, \phi) = \alpha f(\theta, \phi)$.
- If you take one of the eigensolutions, and apply rotations to the arguments, the result will be an eigensolution with the same eigenvalue.
- In fact, the functions Y_l^m form the basis of a vector space. Rotations are represented by matrices that transform the vectors into one another. These matrices form a 'representation of the rotation group'
- The above representation spaces are irreducible – i.e. no subspace is invariant under rotations.
- By separation of variables you can create solutions of the Schrodinger equation of the form

$$\left[-\frac{1}{2mr}\frac{d^2}{dr^2}r + \frac{\mathbf{L}^2}{2mr^2} - \frac{e^2}{r}\right]\chi_l(r)Y_l^m(\theta,\phi) = E_l\chi_l(r)Y_l^m(\theta,\phi). \quad (12)$$

4 Rotation algebra

Here's the question: how many states are there for each energy value? In practice, the Schrodinger equation is modified by terms that slightly perturb

the energies so that they are no longer degenerate. So when we look at the spectrum, we see groupings.

In this section, we use algebraic methods to find the degeneracy of E_l, i.e., the number of energy states for each value of l. Remember, this is the same as asking "how many m values are there for each l"? Or, using fancy language, "what is the dimensionality of the irreducible representation spaces corresponding to 'l'?"

- Define the angular momentum operators

$$\begin{aligned} L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x. \end{aligned} \tag{13}$$

  Remember that the momentum operators represent derivatives, so for example $L_z\psi(x,y,z) = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\psi(x,y,z)$. Although I've gone back to Cartesian coordinates, this can all be converted to spherical coordinates, in which case we can show that $L_z = -i\frac{\partial}{\partial\phi}$.

  With these definitions, we can construct $\mathbf{L}\cdot\mathbf{L} = L_x^2 + L_y^2 + L_z^2$. If you expand this out and convert to spherical coordinates, you discover that this is the expression we had before:

$$\mathbf{L}\cdot\mathbf{L} = -\frac{1}{\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}\right]. \tag{14}$$

  That, of course, is why we called that operator $\mathbf{L}^2$. More generally, we refer to $\mathbf{L}^2$ as a *Casimir operator* for the rotation group. It has the property that its eigenvalues each correspond to an irreducible representation of the rotation group.

- Example with spherical harmonics: Recall that $L_z = -i\frac{\partial}{\partial\phi}$. Apply this operator to the spherical harmonics Y_1^m.

$$\begin{aligned} -i\frac{\partial}{\partial\phi}Y_1^{-1} &= -i\frac{\partial}{\partial\phi}\left(\sqrt{\frac{3}{8\pi}}e^{-i\phi}\sin\theta\right) = -\sqrt{\frac{3}{8\pi}}e^{-i\phi}\sin\theta = (-1)Y_1^{-1} \\ -i\frac{\partial}{\partial\phi}Y_1^{0} &= -i\frac{\partial}{\partial\phi}\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right) = 0 = (0)Y_1^{0} \\ -i\frac{\partial}{\partial\phi}Y_1^{1} &= -i\frac{\partial}{\partial\phi}\left(-\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta\right) = -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta = (1)Y_1^{1}. \end{aligned} \tag{15}$$

We see that these spherical harmonics are eigenfunctions of L_z with eigenvalues $-1, 0$ and 1.

- The angular momentum operators are the *infinitesimal generators of rotations*. Here's how that works. Start with the wavefunction $\psi(\theta, \phi)$. Rotate the coordinates an infinitesimal amount around the z-axis, i.e. $\psi'(\theta, \phi) = \psi(\theta, \phi+\epsilon) = \psi(\theta, \psi)+\epsilon\frac{\partial}{\partial\phi}\psi(\theta, \phi)+... = \psi(\theta, \psi)+i\epsilon L_z\psi(\theta, \phi)+$ .... A similar thing is true for rotations around the other axes. Now, when you first rotate by 2 degrees around the z-axis and then 3 degrees around the x-axis, you get a different result than if you first rotate 3 degrees around the x-axis and then 2 degrees around the z-axis. This implies certain commutation rules for the angular momentum operators.

- By brute force, you can show that the L_i's obey the commutation rule

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{16}$$

  Here, I've identified the indices $(1, 2, 3)$ with (x, y, z). These commutation relations are intrinsic **properties of the rotation group** and can be derived directly from the matrix operations that define vector rotations.

- Abstractly, a scalar operator (in non-relativistic mechanics, the Hamiltonian is a scalar operator), $\mathbf{O}$, is symmetric under rotations, if and only if $\mathbf{ROR}^{-1} = \mathbf{O}$ for any rotation, and thus if and only if $[L_i, \mathbf{O}] = 0$ for each i, since the L_i generate rotations.

- **IMPORTANT ALGEBRAIC RESULT 1** (Exercise)

$$[L_i, \mathbf{L}^2] = 0 \tag{17}$$

  This is an expression of the rotational invariance of the operator $\mathbf{L}^2$.

- **IMPORTANT ALGEBRAIC RESULT 2**. Suppose f is an eigenfunction of $\mathbf{L}^2$ with eigenvalue α, i.e. $\mathbf{L}^2 f_\alpha(\theta, \phi) = \alpha f_\alpha(\theta, \phi)$. Then if f'_α is the result of operating by rotation on the arguments of f_α (i.e. $f'_\alpha(\theta, \phi) = f_\alpha(R(\theta, \phi))$), then f'_α is also an eigenfunction of $\mathbf{L}^2$ with eigenvalue α. **This is the critical important origin of energy degeneracy!**. Proof: (important method) The starting assumption is that

$$\mathbf{L}^2 f_\alpha = \alpha f_\alpha \tag{18}$$

Instead of deriving the proof for general rotations $\mathbf{R}$, it suffices to derive the proof for the generators of rotations L_i. So consider the new function $L_i f_\alpha$ and apply the operator $\mathbf{L}^2$.

$$\begin{aligned} \mathbf{L}^2(L_i f_\alpha) &= L_i(\mathbf{L}^2 f_\alpha) \\ &= L_i(\alpha f_\alpha) \\ &= \alpha(L_i f_\alpha), \end{aligned} \tag{19}$$

where the first line comes from the commutation of L_i and $\mathbf{L}^2$, the second line comes from the original eigenvalue condition, and the final line comes from the fact that numbers like α commute with all operators. The net result is that $L_i f_\alpha$ is an eigenfunction of $\mathbf{L}^2$ with eigenvalue α. Q.E.D.

- **MAJOR ALGEBRAIC RESULT**. We come to the calculation of multiplicity. This technique generalizes to all (compact Lie) symmetry groups!

  Theorem: There are a total of $2l+1$ independent eigenfunctions with $\mathbf{L}^2 f_\alpha(\theta,\phi) = l(l+1) f_\alpha(\theta,\phi)$. These eigenfunctions can be chosen to be eigenfunctions of the operator L_z with integer eigenvalues m so that $|m| \leq l$.

  Corollary 1:The dimension of the irreducible representation space corresponding to the eigenvalue l, is $2l+1$.

  Corollary 2: The spherical harmonics Y_l^m have values of m ranging from $-l$ to l and these span a $2l+1$ irreducible representation space. In particular, this gives us a way of constructing irreducible representations for all odd dimensions.

  Lemma 1: Define $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y = L_+^\dagger$. Then $L_+ L_-$ is a positive self-adjoint operator with positive or 0 eigenvalues.

  Proof of Lemma 1: For any operator $\mathbf{O}$, the new operator $\mathbf{O}\mathbf{O}^\dagger$ is both positive and self-adjoint.

  Lemma 2: For each value of l, there is at least one eigenfunction ψ of L_z so that $\mathbf{L}^2\psi = l(l+1)\psi$ and $L_z\psi = \beta\psi$.

  Proof of Lemma 2: I'm not going to prove the first part of this – namely, that there is always at least one eigenfunction whose $\mathbf{L}^2$ eigenvalue is $l(l+1)$. Throughout this presentation, this has been an assumption and it was claimed that the spherical harmonics were such

functions.[1] However, since L_z is a self-adjoint operator (generating rotations around the z-axis) acting in the representation space, it must have an eigenfunction within that space, by general properties of self-adjoint operators.

Lemma 3: If $L_z\psi = \beta\psi$, then $L_z(L_+\psi) = (\beta+1)(L_+\psi)$. Hence $L_+\psi$ is an eigenfunction of L_z whose eigenvalue is $\beta+1$ **UNLESS** $L_+\psi = 0$**!**. We call L_+ a raising operator. Similarly, $L_z(L_-\psi) = (\beta-1)(L_-\psi)$ and L_- is called a lowering operator.

Proof of Lemma 3: **This is one of the bread and butter proofs used frequently in QM.** First observe (exercise) that

$$\begin{aligned} [L_z, L_+] &= L_+ \\ [L_z, L_-] &= L_- \end{aligned} \tag{20}$$

Then

$$\begin{aligned} L_z(L_+\psi) = (L_+L_z + L_+)\psi &= L_+(L_z\psi) + L_+\psi \\ &= L_+(\beta\psi) + L_+\psi \\ &= L_+(\beta+1)\psi \\ &= (\beta+1)L_+\psi. \end{aligned} \tag{21}$$

The first line expresses the commutation relation of the previous equation (20). The second line expresses the assumption that ψ is an L_z eigenfunction with eigenvalue β. The final line is Q.E.D. A similar proof can be used to show that L_- is a lowering operator.

Lemma 4: With the same eigenfunction ψ as above, $L_-L_+\psi = [l(l+1) - \beta(\beta+1)]\,\psi$ and $L_+L_-\psi = [l(l+1) - \beta(\beta-1)]\,\psi$.

Proof of Lemma 4: First notice that $L_-L_+ = \mathbf{L}^2 - L_z^2 - L_z$. We derive this from

$$\begin{aligned} L_-L_+ &= (L_x - iL_y)(L_x + iL_y) \\ &= L_x^2 + L_y^2 - iL_yL_x + iL_xL_y \\ &= \left(\mathbf{L}^2 - L_z^2\right) - L_z \end{aligned} \tag{22}$$

The last term follows from the commutation relation $[L_y, L_x] = -iL_z$.

[1] For more general compact Lie groups, we start by proving that there are finite-dimensional irreducible representations, and that one can construct Casimir operators like $\mathbf{L}^2$ whose eigenvalues characterize the dimension of the irreducible spaces.

Now we apply the operator L_-L_+ to ψ. From above, we have

$$\begin{aligned} L_-L_+\psi &= \left[\mathbf{L}^2 - L_z^2 - L_z\right]\psi \\ &= \left[l(l+1) - \beta^2 - \beta\right]\psi \\ &= \left[l(l+1) - \beta(\beta+1)\right]\psi \end{aligned} \tag{23}$$

Similarly for L_+L_-. Q.E.D.

Lemma 5: We must have $|\beta| \leq l$.

Proof of Lemma 5: Otherwise, if $\beta \geq 0$ then the eigenvalue of L_-L_+ would be negative, or if $\beta \leq 0$ then the eigenvalue of L_+L_- would be negative, which would contradict Lemma 1.

Lemma 6: When $l = 1$, there are exactly 3 distinct eigenvalues of L_z. Those values are $\beta = -1, 0, 1$.

Proof of Lemma 6: The steps below are illustrative of the proof for general values of l.

- The following illustrates what goes wrong if one of the eigenvalues is fractional, for example suppose $\beta = \frac{1}{2}$. By assumption, we have an eigenfunction $\psi_{\frac{1}{2}}$ so that $L_z\psi_{\frac{1}{2}} = \frac{1}{2}\psi_{\frac{1}{2}}$.
- If we apply the raising operator L_+ to $\psi_{\frac{1}{2}}$, one of two things could happen. Either the result could be 0. Or we would have a new eigenfunction whose L_z eigenvalue is $\frac{3}{2}$. But in that case, the eigenvalue is greater than 1 so contradicts Lemma 6. Therefore we've proven the first alternative, $L_+\psi_{\frac{1}{2}} = 0$.
- Now apply L_- to $\psi_{\frac{1}{2}}$. We'll show that this is **not** 0. Consider the operator L_+L_- acting on $\psi_{\frac{1}{2}}$. From Lemma 4

$$\begin{aligned} L_+L_-\psi_{\frac{1}{2}} &= \left[l(l+1) - \beta(\beta-1)\right]\psi_{\frac{1}{2}} \\ &= \frac{9}{4}\psi_{\frac{1}{2}} \end{aligned} \tag{24}$$

  What we've proven is that $L_+L_-\psi_{\frac{1}{2}} \neq 0$, and therefore $L_-\psi_{\frac{1}{2}} \neq 0$ as claimed. Define $\psi_{-\frac{1}{2}} = L_-\psi_{\frac{1}{2}}$.
- Since L_- is a lowering operator, we know that $L_z\psi_{-\frac{1}{2}} = (\frac{1}{2} - 1)\psi_{-\frac{1}{2}} = (-\frac{1}{2})\psi_{-\frac{1}{2}}$.

– Now repeat the process by applying L_- to $\psi_{-\frac{1}{2}}$. We will show this is not 0 and therefore that we will have created a new (nonzero) vector whose eigenvalue is $-\frac{1}{2}-1$. But from Lemma 5, this isn't possible. So our starting premise must have been wrong. We proceed as before, by showing $L_+L_-\psi_{-\frac{1}{2}} \neq 0$.

$$\begin{aligned} L_+L_-\psi_{-\frac{1}{2}} &= \left[l(l+1) - \left(-\frac{1}{2}(-\frac{1}{2}-1)\right)\right]\psi_{-\frac{1}{2}} \\ &= \frac{5}{4}\psi_{-\frac{1}{2}} \neq 0 \end{aligned} \tag{25}$$

– What we've shown above, is that if we assume that L_z has an eigenvalue of $\frac{1}{2}$ we obtain a contradiction. A similar contradiction can be easily proven, in the same way, for any fractional eigenvalue of L_z.

  The largest possible eigenvalue, based on Lemma 5, would be 1. If we start by assuming a non-zero eigenvector ψ_1 such that $L_z\psi_1 = \psi_1$, then by following the procedure of the past few steps, we can show there are non-zero eigenfunctions ψ_0 and ψ_{-1} whose L_z eigenvalues are respectively 0 and -1.

– The only thing remaining to be shown to prove Lemma 6, is what happens if we start with, for example, an eigenfunction ψ_0 defined by $L_z\psi_0 = 0$. (We could instead have started with an eigenfunction ψ_-1, using the same methodology that follows.) We would prove, first of all, that $L_+\psi_0 \neq 0$. We would do this in a similar fashion to above, by showing that $L_-L_+\psi_0 \neq 0$, employing Lemma 4 for L_-L_+. Then, since L_+ is a raising operator, we know that the nonzero function $L_+\psi_0$ would have L_z eigenvalue of 1. That would bring us back to the starting point used in earlier steps where we showed that there were three eigenfunctions of L_z.

Proof of Theorem: Lemma 6 was a proof of the theorem for the case that $l = 1$. The same procedure can be used to generalize to any value of l. A systematic method is this: Lemma 5 shows that the eigenvalues fall in the range $-l \to l$, so pick an eigenfunction with an eigenvalue in that range. Apply the raising operator, and check (using Lemma 4) whether the result is 0. If not, raise it again, noting that with each 'raise', the L_z eigenvalue goes up by 1. Eventually, you get

to a maximum value (remembering that Lemma 5 prevents us from going on forever). Once you reach that eigenfunction, it becomes the new starting point where you systematically apply to it the lowering operator, each time verifying (using Lemma 4) that you get a nonzero value. If your starting point has the eigenvalue $\beta = l$, then you'll be able to continue all the way to $\beta = -l$. Otherwise, you'll be starting with a fractional eigenvalue (remember, the eigenvalue needs to satisfy the inequality $l - 1 < \beta \leq l$) and you'll eventually create a non-zero state whose eigenvalue is less than $-l$, something prohibited by Lemma 5. Therefore you'll have shown that the only valid situation is one where the eigenvalues are integers inclusively between $-l$ and l. Q.E.D.

5 EXERCISES

1. Prove that $[L_z, \mathbf{L}^2] = 0$.

   Hint:

$$\begin{aligned} L_z(L_xL_x) &= (L_zL_x)L_x \\ &= (L_xL_z - iL_y)L_x \\ &= (L_xL_z)L_x - i(L_yL_x) \\ &= L_x(L_zL_x) - i(L_yL_x) \\ &= L_x(L_xL_z - iL_y) - i(L_yL_x) \\ &= (L_xL_x)L_z - i(L_xL_y + L_yL_x) \end{aligned} \tag{26}$$

2. Prove that $[L_z, L_+] = L_+$.

   Expand L_+ and then use the L_i commutation relations.