Fields and particles – what they don't tell you in most QFT books

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Physicists in the 60's, exploring QFT in a background (gravitational) metric, became concerned that the well-known correspondence between fields and particles, was ambiguous if spacetime was arbitrarily curved. The nature of this ambiguity and its interpretation, was resolved by Bill Unruh, who noted two things. First, the ambiguity was already present in ordinary Minkowski space and could be made manifest in the fashion to be outlined below. And second, the ambiguity really wasn't an issue. Rather, as Unruh explained, it was necessary to discuss how particles were measured. He proposed a gedanken experiment in which particles were counted by the clicks of a simplified geiger counter (which is called, in the GR literature, an Unruh detector). Suppose you have an environment (which you will call, for the sake of a name, the 'vacuum') defined so that, in an inertial reference frame, the geiger counter registers no counts (there are no particles). Now take the same geiger counter and accelerate it. From the point of view of the geiger counter, using accelerated coordinates, it is at rest (although experiencing non-inertial forces) but it turns out that the geiger counter is clicking away like crazy. So, in the accelerated coordinates there appear to be lots of particles, but in the inertial frame there are none. There is no contradiction here. An accelerating geiger counter is a different beast than one at rest. In order for the acceleration to occur, a force must be applied to the geiger counter. This force, when described on a microscopic level, has the effect of changing the vacuum to something different. So what Unruh showed, is that the particle concept is meaningful only in the context of a specified measurement, and that the measuring apparatus might care about whether or not it is in an inertial frame. Unruh then turned to the original issue – particle ambiguities in the presence of a gravitational field. He pointed out that the situation was very similar to the one in Minkowski space with accelerated coordinates, and that the particle-ambiguity was a perfectly understandable consequence of coordinate choice. What makes the gravitational situation a bit different in nature, is what one chooses to describe as 'natural coordinates'. The inertial coordinates aren't 'natural' since they don't correspond to the coordinates we use to describe our environment and also, because in some (many) cases, there are no universal inertial coordinates.

As a result of all this, one has a situation where gravity appears to create particles out of the vacuum. However, that statement is too facile and really amounts to everything above.

1 References

There are four references that I use extensively:

- N.D.Birrell and P.C.W. Davies Quantum fields in curved space. This is my primary reference and I mostly follow that notation. It is quite mathematical and complete and has a huge number of bibliographical references to fill in many blanks. Some blanks are bigger than others and I'll mention them if and when appropriate.

- Sean M. Carroll Spacetime and Geometry. In many ways this is more readable than Birrell and Davies, but is much less comprehensive. In particular, although it does an excellent job of explaining particle ambiguities, the explanation may require somewhat more context than mine below (which borrows from Birrell and Davies).

- Bryce S. DeWitt, "Quantum Theory of Gravity I" published in Phys. Rev. 160 volume 25 page 1113 (1967). This is the Bible and almost all modern discussions of Quantum Gravity end up referring back to this paper and two others written by DeWitt at that time (really it was one huge paper broken into three parts). Ultimately, if you want to dive into areas requiring rigorous analysis, and to properly understand the way to develop a quantum theory of gravity, I think this paper is essential. Having said that, it's a huge amount of work to go through it carefully or even quickly. I haven't done that work and tend to rely on second or third hand accounts of the contents.

- Robert M. Wald General Relativity in Chapter 14 "Quantum Effects in Strong Gravitational Fields". Like Carroll's book, the section on quantum mechanics is short and the rest of the text covers only the non-quantum theory of gravity. Wald's chapter is, in my opinion, truly remarkable. The treatment – like the rest of his book – is mathematically solid. Wald also manages to do a an excellent job exposing the theoretical difficulties of combining GR with QM. I was hoping to find, in Wald, a treatment of the canonical commutation relations in a general background metric (this is something that I use in my notes below). That's a topic that often comes up in the literature, including some of the above references. However, I think Wald has managed to get around it as follows (I don't guarantee I've analyzed this well enough so don't take my word for it): In gravitational theories of interest, the far reaches of the universe (far past and far future, as well as current time but distant space) tend towards flatness – i.e., Minkowski space. We know how to quantize theories in Minkowski space, so we can apply canonical commutation operator relations in the far reaches of the universe. Evidently, that suffices for establishing an S-matrix that describes how things evolve from the far past to the far future. So if that's true, then there's no need to say anything about how the operators commute at intermediate times.

I've also encountered an article which addresses the entirety of the issues covered below. "A note on canonical quantization of fields on a manifold" by Moschella and Schaeffer "https://arxiv.org/pdf/0802.2447.pdf ". I believe my notes below follow more closely the material on QFT which may be most familiar to the readers of the majority of QFT textbooks.

2 Particle ambiguity in 2D Minkowski space

2.1 The theory to be examined in this note

For simplicity, we will examine a massless scalar theory, without any detector. In the usual Minkowski coordinates, the Lagrangian of the theory is

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x). \tag{1}$$

2.2 Two ways to construct a Hilbert space

There are at least two approaches to constructing a quantum theory. One way, with which I have the most experience and will follow below, is to start by using properties of the operators, based on commutation relations and the equations of motion for the fields. From this, one can instantiate a Hilbert space of states so that the fields operate on those states in a way that leads to the right commutation operators. I will refer to this approach as the *canonical method*.

Another approach, described for example in Wald, is to start with the space of complex functions that solve the equations of motion. A certain inner product (known as the Klein-Gordan inner product) can be set up on that space, a basis can be found for that Hilbert space, and finally a Fock space basis can be created out of the above basis states. In this approach, we complete the procedure by hypothesizing that the field operators are a particular linear combination of basic operators (the annihilation and creation operators) that have simple behaviors when applied to the Fock basis. Commutation relations are an outcome of these definitions, but are not asserted as axioms. I believe Wightman's axiomatic approach to field theory may be closer in spirit to Wald's. The two approaches (canonical and Wald) appear to be equivalent.

2.3 Canonical method – Inertial coordinates

- The wave equation (obtained by finding the extremum of the action) is

$$\partial_\mu \partial^\mu \phi(t, x) = 0, \tag{2}$$

  where the index μ takes on the value 0 or 1.

- When we promote fields to operators (on a yet TBD Hilbert space), the canonical commutation relations[1] become

$$\begin{aligned} [\phi(t,x), \phi(t,x^{'})] &= 0 \\ [\partial_t \phi(t,x), \partial_t \phi(t,x^{'})] &= 0 \\ [\phi(t,x), \partial_t \phi(t,x^{'})] &= i\delta(x - x^{'}). \end{aligned} \tag{4}$$

  Notice that form of these commutation relations don't appear to be Lorentz symmetric. If there are two spacial points appearing at the same time in one reference frame, then in a different reference frame they are likely to be at different times. So equal-time commutation relations in one frame aren't equal-time in a different frame. A question which must be addressed for the sake of a consistent Lorentz invariant theory, is whether, under a Lorentz transformation of coordinates, the fields continue to obey equal-time commutation relations in the transformed coordinates. The answer is 'yes'. I'll return to that shortly.

[1]

$$\begin{aligned} [\phi_i(t,x), \Pi_j(t,x^{'})] &= i\delta_{ij}\delta(x - x^{'}) \\ [\phi_i(t,x), \phi_j(t,x^{'})] = [\Pi_i(t,x), \Pi_j(t,x^{'})] &= 0 \end{aligned} \tag{3}$$

- A general solution to the wave equation is

$$\phi(t,x) = \sum_k \left[a_k e^{i(k_1 x - \omega_k t)} + a_k^\dagger e^{-i(k_1 x - i\omega_k t)} \right], \tag{5}$$

where $\omega_{\mathbf{k}} = |k|$, a_k and $a_k^\dagger$ are operators (on the same TBD Hilbert space as above), and the sum over k should be regarded as an integral with the appropriate measure $\frac{dk}{(2\pi)\sqrt{2\omega_k}}$ (or alternatively, if the theory were quantized in a box, then the momenta would indeed by discrete).

- Since the above equation is a Fourier transform, we can easily expand out and simplify the commutation relations (eq. (4)) to become

$$\begin{aligned} [a_k, a_{k'}] &= 0 \\ [a_k^\dagger, a_{k'}^\dagger] &= 0 \\ [a_k, a_{k'}^\dagger] &= \delta_{kk'} \end{aligned} \tag{6}$$

- Assume that there is a (unique) vacuum state (or, equivalently, assume the Hamiltonian is bounded from below – but that assumption requires more explanation) defined as the state $|\Omega\rangle$ with the property

$$a_k |\Omega >= 0, \forall k. \tag{7}$$

Define the states

$$|{}^1n_{k_1}, {}^2n_{k_2}, ..., {}^j n_{k_j}\rangle = \left({}^1n!{}^2n!...{}^jn!\right)^{-\frac{1}{2}} (a_{k_1}^\dagger)^{{}^1n} (a_{k_2}^\dagger)^{{}^2n} ... (a_{k_1}^\dagger)^{{}^jn} |\Omega > \tag{8}$$

These are interpreted as many-particle states, and they form a Fock space – which is the Hilbert space on which the field operators act.

- The remaining part of this section will address an issue raised earlier – the form of the equal-time commutation relations in a different Lorentz frame. **It's safe to skip this discussion**, since the result will be that the fields satisfy equal-time commutation relations in all inertial frames (those related by Lorentz transformations). To prove this, we observe that we can use the wave solution eq. (5) to construct $[\phi(t,x), \phi(t', x^{'})]$ which extends the equal-time commutator to unequal times. By using both eq. (5) and the annihilation-creation relations of eq. (6) we can show (see, for example, Birrell and Davies) that

$$[\phi(t,x), \phi(t', x^{'})] = iG\left((t,x), (t'x')\right) \tag{9}$$

where the Schwinger function $G\left((t,x),(t'x')\right)$ is defined as

$$G\left((t,x),(t'x')\right) = \frac{1}{2\pi}\int dk^0 dk^1 \frac{e^{ik^1(x-x^{'})-ik^0(t-t^{'})}}{(k^0)^2-|k^1|^2} \tag{10}$$

and the k^0 integral is a complex contour clockwise-encircling poles at $k^0 = \pm|k^1|$.

- Now transform coordinates from (t,x) to $(\tilde{t},\tilde{x})$ so that $(t,x) = (\Lambda_{00}\tilde{t} + \Lambda_{01}\tilde{x}, \Lambda_{10}\tilde{t} + \Lambda_{11}\tilde{x})$,[2] . We obtain

$$\begin{gathered} G\left((t,x),(t'x')\right) = \tilde{G}\left((\tilde{t},\tilde{x}),(\tilde{t}',\tilde{x}')\right) = \\ \frac{1}{2\pi}\int dk^0 dk^1 \frac{e^{ik^1\Lambda_{11}(\tilde{x}-\tilde{x}^{'})-ik^0\Lambda_{01}(\tilde{x}-\tilde{x}^{'})}e^{ik^1\Lambda_{10}(\tilde{t}-\tilde{t}')-ik^0\Lambda_{00}(\tilde{t}-\tilde{t}')}}{(k^0)^2-|k^1|^2} \end{gathered} \tag{11}$$

We perform the contour integral and take the residues at the two poles.

$$\begin{aligned} \tilde{G}\left((\tilde{t},\tilde{x}),(\tilde{t}',\tilde{x}')\right) = -i\int \frac{dk^1}{2k^1} e^{ik^1\left[(\Lambda_{11}-\Lambda_{01})(\tilde{x}-\tilde{x}')+(\Lambda_{10}-\Lambda_{00})(\tilde{t}-\tilde{t}')\right]} \\ + i\int \frac{dk^1}{2k^1} e^{ik^1\left[(\Lambda_{11}+\Lambda_{01})(\tilde{x}-\tilde{x}')+(\Lambda_{10}+\Lambda_{00})(\tilde{t}-\tilde{t}')\right]} \end{aligned} \tag{12}$$

Notice that the individual integrals diverge at $k^1 = 0$, but when combined, the divergences disappear (expand the exponentials to see how this works).

- We can use this expression to examine the commutation relations in the new coordinates. Note that $\tilde{\phi}(\tilde{t},\tilde{x}) \equiv \phi(t,x)$. Then, using our expression eq. (12) for the Schwinger function when $\tilde{t} = \tilde{t}'$,

$$[\tilde{\phi}(\tilde{t},\tilde{x}), \tilde{\phi}(\tilde{t},\tilde{x}')] = i\tilde{G}\left((\tilde{t},\tilde{x}),(\tilde{t},\tilde{x}')\right) \quad = 0 \tag{13}$$

I don't know a truly simple (but rigorous) way of proving this. A straightforward, but seemingly overly complicated, derivation goes as follows: If we define the Fourier transform (involving $(\tilde{x}-\tilde{x}')$) as an integral from $-A$ to A for non-zero values of $(\tilde{x}-\tilde{x}')$, then take the limit $A \to \infty$ before taking the limit $(\tilde{x}-\tilde{x}') \to 0$, it can be shown by contour integration and explicit integrals, that the result is 0, provided that $|\Lambda_{11}| > |\Lambda_{10}|$. This inequality is satisfied for all Lorentz transformations. A more thorough discussion of these integrals can be found in treatments, such as in Schwartz, of the proof of causality in QFT.

[2]It's useful to know, when proving Λ-matrix identities, that for some value of β, the matrix elements can be rewritten as $\Lambda_{00} = \Lambda_{11} = \cosh\beta$ and $\Lambda_{01} = \Lambda_{10} = \sinh\beta$.

- We can also compute

$$
\begin{aligned}
[\tilde{\phi}(\tilde{t},\tilde{x}), \partial_{\tilde{t}'}\tilde{\phi}(\tilde{t}',\tilde{x}')|_{\tilde{t}'=\tilde{t}}] &= i\partial_{\tilde{t}'}\tilde{G}\left((\tilde{t},\tilde{x}),(\tilde{t},\tilde{x}')\right)|_{\tilde{t}'=\tilde{t}} \\
&= -\frac{i}{2}\int dk^1\left[(\Lambda_{10}-\Lambda_{00})e^{ik^1(\Lambda_{11}-\Lambda_{01})(\tilde{x}-\tilde{x}')}\right]+ \\
&\quad \frac{i}{2}\int dk^1\left[(\Lambda_{10}+\Lambda_{00})e^{ik^1(\Lambda_{11}+\Lambda_{01})(\tilde{x}-\tilde{x}')}\right] \\
&= -\frac{i}{2}\left[\frac{\Lambda_{10}-\Lambda_{00}}{\Lambda_{11}-\Lambda_{01}}-\frac{\Lambda_{10}+\Lambda_{00}}{\Lambda_{11}+\Lambda_{01}}\right]\delta(\tilde{x}-\tilde{x}') \\
&= i\delta(\tilde{x}-\tilde{x}').
\end{aligned} \tag{14}
$$

The above was derived by changing the integration variable to obtain the Fourier transform of the delta function, and then using the properties of Lorentz transformations. This expression demonstrates the second of the equal-time (for $\tilde{t}$) commutation relations in eq. (4).

- The remaining commutation relation to be examined is the equal time commutator of the canonical momentum operators (the third equation of eq. (4)). This follows closely the procedure of the second commutator.

$$
\begin{aligned}
[\partial_{\tilde{t}}\tilde{\phi}(\tilde{t},\tilde{x}), \partial_{\tilde{t}'}\tilde{\phi}(\tilde{t}',\tilde{x}')|_{\tilde{t}'=\tilde{t}}] &= i\partial_{\tilde{t}}\partial_{\tilde{t}'}\tilde{G}\left((\tilde{t},\tilde{x}),(\tilde{t},\tilde{x}')\right)|_{\tilde{t}'=\tilde{t}} \\
&= \frac{1}{2}\int k^1 dk^1\left[(\Lambda_{10}-\Lambda_{00})^2 e^{ik^1(\Lambda_{11}-\Lambda_{01})(\tilde{x}-\tilde{x}')}\right]- \\
&\quad \frac{1}{2}\int k^1 dk^1\left[(\Lambda_{10}+\Lambda_{00})^2 e^{ik^1(\Lambda_{11}+\Lambda_{01})(\tilde{x}-\tilde{x}')}\right] \\
&= -\frac{i}{2}\left[\left(\frac{\Lambda_{10}-\Lambda_{00}}{\Lambda_{11}-\Lambda_{01}}\right)^2-\left(\frac{\Lambda_{10}+\Lambda_{00}}{\Lambda_{11}+\Lambda_{01}}\right)^2\right]\partial_{\tilde{x}}\delta(\tilde{x}-\tilde{x}') \\
&= 0,
\end{aligned} \tag{15}
$$

where the last equation is a result of Λ-matrix identities.

2.4 Canonical method – Accelerated coordinates

Is the above Fock space the only reasonable construction and interpretation of a Hilbert space on which the field operators can act? I'll now proceed to show that it is not. But before doing so, it's important to realize that the conclusion is important and the method isn't. The conclusion is an illustration of the following statement. There isn't a unique basis for the infinite dimensional space of solutions to the equation of motions, and therefore there is not a unique expansion of the field in terms of basis functions.

What follows, is an example of the non-uniqueness of decomposition. In more comprehensive treatments of the subject, including Unruh's, further work is done in order to establish a connection between particular Fock-space constructions and how these describe, in a measurable way, the familiar properties of particles. Later, I'll touch briefly on the Unruh discussion.

We begin our example by writing the theory in terms of a different set of coordinates. Then we show that we can construct a Fock space for that rewritten theory, using an almost identical procedure to what we did in inertial coordinates. However, that Fock space will be different, and in particular, its vacuum is **not** the vacuum of the inertial-coordinate theory.

The coordinates I'll use below are known as Rindler coordinates. The literature often discusses how these coordinates should be regarded, but I don't think that's necessary for any of the points to be made in this note.

- The Rindler coordinates are defined by:

$$\begin{aligned} t &= a^{-1} e^{a\xi} \sinh a\eta \\ x &= a^{-1} e^{a\xi} \cosh a\eta \end{aligned} \tag{16}$$

  where a = constant > 0, $-\infty < \eta, \xi < \infty$. The coordinates (η, ξ) only cover a quadrant of Minkowski space ($x > |t|$). This quadrant is conventionally designated as **R**. Alternatively, we can define η and ξ by changing the sign of the RHS of the above coordinate transformations, in which case a different quadrant, **L**, (($x < |t|$)) of Minkowski space is covered.

- By transforming coordinates for eq. (5), we obtain for the transformed field $\tilde{\phi}$

$$\tilde{\phi}(\eta, \xi) = \sum_{\mathbf{k}} \left[a_{\mathbf{k}} e^{ik_0 a^{-1} e^{a\xi} \sinh a\eta - ik_1 a^{-1} e^{a\xi} \cosh a\eta} + a_{\mathbf{k}}^{\dagger} e^{-ik_0 a^{-1} e^{a\xi} \sinh a\eta + ik_1 a^{-1} e^{a\xi} \cosh a\eta} \right] \tag{17}$$

  Now, this is a perfectly valid expansion. However, it isn't the only valid expansion.

- By explicitly changing the coordinates using eq. (16), we find that the following equation holds for $\tilde{\phi}$.

$$\left(\frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \xi^2} \right) \tilde{\phi} = 0. \tag{18}$$

  This can be solved as a Fourier transform, just as we did in the inertial coordinates. However, there is a subtlety. The field $\tilde{\phi}$ is an expression for the field **only** in the quadrant **R** of Minkowski space. If we want an expression of physics that covers all of Minkowski space, we need coordinates that can be used for the entire space. As mentioned earlier, an alternate set of coordinates, based on changing the sign of the RHS of eq. (16), can be used to cover the quadrant **L** of Minkowski space. We can steer clear of any ambiguity by using different coordinate-notation for the two quadrants. Alternatively, and more commonly done by physicists, we can capture the same idea as follows. We define basis functions

$$\begin{aligned} {}^{R}u_k &= \begin{cases} (4\pi\omega)^{-\frac{1}{2}} e^{ik\xi - i\omega\eta}, & \text{in } \mathbf{R} \\ 0, & \text{in } \mathbf{L} \end{cases} \\ {}^{L}u_k &= \begin{cases} (4\pi\omega)^{-\frac{1}{2}} e^{ik\xi + i\omega\eta}, & \text{in } \mathbf{L} \\ 0, & \text{in } \mathbf{R} \end{cases} \end{aligned} \tag{19}$$

These, together, form Fourier bases so that if we change the Rindler coordinates (defined properly for each of **L** and **R**) to inertial coordinates, it can be seen that the domains of the basis functions together cover **L** and **R**. In particular, the entire η axis is included and comprises a Cauchy surface for the space. [3]

- The new Fourier basis functions look similar to the old ones, but have the property that they need to be 'glued' together to cover the entire domain. The general solution looks like

$$\tilde{\phi} = \sum_{k=-\infty}^{k=\infty} \left[b_{\mathbf{k}}^{(1)}({}^{L}u_k) + b_{\mathbf{k}}^{(1)\dagger}({}^{L}u_k^*) + b_{\mathbf{k}}^{(2)}({}^{R}u_k) + b_{\mathbf{k}}^{(2)\dagger}({}^{R}u_k^*) \right] \tag{20}$$

  where the b_k and $b_k^\dagger$ are operators.

- Now impose the equal-η (analogous to equal-time) canonical commutation rules on $\tilde{\phi}$ and $\partial_\eta \tilde{\phi}$. Earlier in this note, I pointed out that it

[3] I think that this discussion regarding the requirement for two sets of basis functions – one for both the **L** and **R** quadrants – can't really be understood except by going through the analysis of how one set of basis functions (and especially their coefficients in the expansion) can be expanded in terms of the other set of basis functions. It's reasonable to imagine that at the very least, both sets of basis functions would be defined on the η axis. On the other hand, one might then wonder why it wouldn't be necessary for Rindler-style basis functions to be defined on all 4 quadrants. What is said in Birrell and Davies, is that by analytic continuation of the basis functions (specifically by going to imaginary values of a), the remaining two quadrants are also covered. Perhaps (one of these days I might look into this) the point is that the coefficients can be uniquely expanded in terms of one another, based only on the values of the basis functions within the **L** and **R** quadrants.

wasn't obvious (at least to me) that these commutation relations apply in the new (Rindler) coordinates. In fact, it was even somewhat nontrivial to show that the equal-time commutators were independent of inertial frame. In the appendix, I will (partly) extend these conclusions to Rindler coordinates. For now, assume those commutation relations. Then by performing Fourier transforms as we did in the inertial coordinates, we can show that

$$[b_k^{(1)}, b_{k'}^{(1)\dagger}] = [b_{\mathbf{k}}^{(2)}, b_{k'}^{(2)\dagger}] = \delta_{kk'} \tag{21}$$

with all other combinations commuting.

- Then just as we did for inertial coordinates, we can find a state $|\Omega\rangle_L$ annihilated by $b_k^{(1)}$ and a state $|\Omega\rangle_R$ annihilated by $b_k^{(2)}$. We say that the Rindler vacuum $|\Omega >_{\text{Rindler}} = |\Omega\rangle_L |\Omega\rangle_R$. We then proceed to build the Fock space by applying the creation operators $b_k^{(1)\dagger}$ and $b_k^{(2)\dagger}$.
- What we can see in comparing eqs. (17) and (20) is that the a_k and $b_k^{(i)}$ operators may be different, and therefore the inertial vacuum Ω may be different than the Rindler vacuum $|\Omega >_{\text{Rindler}}$. To actually prove that the two vacua are different, requires some further analysis of the basis states and their analyticity properties in each of the coordinate systems. One of the key aspects of this proof, is the expansion of operators like $b_k^{(1)}$ in terms of a_k and $a_k^\dagger$. If the expansion of a Rindler annihilation operator involves an inertial creation operator, then the vacua are different. These expansions depend on the expansion of the Rindler basis functions in terms of $e^{ik(x-t)}$ etc. This, in turn, will require some care in matching the various basis functions at the intersection between regions **L** and **R**, so that their sums have the same analyticity and boundedness properties as $e^{ik(x-t)}$ and so on.
- The point of this note was to illustrate how it's possible for the same Lagrangian theory to result in (at least) two different annihilation-creation expansions for the field operators where the vacuum (and one-particle states) of one set of annihilation-creation operators, are different states of the Fock space than the vacuum (and one-particle states) of the other set of annihilation-creation operators.
- The remaining question is "which of the field-operator expansions should we use?" It's tempting to think one should use whichever corresponds to the coordinate system where the observer is at rest. Certainly, in that coordinate system the expansion basis functions look the simplest.

(At least, this is true for the particular sets of coordinates that we have explored.) Still, simplicity isn't necessarily a relevant criterion. It would only be relevant if, in the chosen coordinate system, physical behaviors – as predicted from the field theory – are the ones we'd expect. What Unruh examined in order to properly define the concept of particles, was the behavior of a geiger counter at rest with respect to whatever coordinate system is being considered. The geiger counter counts the number of particles in its environment. The probability of a count being registered, is proportional to an overlap (magnitude-squared of the inner-product) between the initial state of the field and its one-particle states (which induce a detector transition) and since the initial field state is the vacuum (no overlap with one-particle states), the Geiger counter registers no counts. Now we might have guessed that in this vacuum state, the detector would continue to register no counts even when the detector is accelerating and at rest in the Rindler coordinates. However, the analysis above shows that in the Rindler coordinate system, the initial field state, which is an inertial vacuum state, corresponds to a Rindler non-vacuum state and therefore there is a non-zero overlap between the initial (Rindler) state and one-particle (Rindler) states. So the Geiger counter registers some counts. If we insist on interpreting the Geiger counter's events as 'particle counts', then we would come to the conclusion that the meaning of 'particle' is coordinate-independent. And, in general, that turns out to be true.

2.5 Curved spacetime

Here's a brief word about generalizing to curved spacetime. So far, the example above only pertained to Minkowski spacetime, with several different sets of coordinate systems. We were able to start with the commutation relations in inertial coordinates and derive everything else from those. As it happened, we saw that in Rindler coordinates, the commutation relations appear similar to those in inertial coordinates.

When we consider curved spacetime, it generally isn't possible to transform to a set of coordinates that are globally Minkowski. Therefore we don't have a starting point with known commutation relations. It becomes necessary to postulate an appropriate set of commutation relations in at least one coordinate system. Since this is a new postulate, there is some freedom in how to proceed. By now, a well-established procedure has been in place for about 60 years. The procedure has been analyzed to death and many alternatives have been compared (and generally found to be equivalent). The Hilbert space constructions follow similar lines to what was done above, and the particle-interpretation ambiguities occur for the same reasons. Owing to the work of Unruh and others, those ambiguities are reasonably understood to be resolved by appropriate specifications of what observations are being analyzed. Nevertheless, the general subject leads to the so-called information paradox, which amongst other things is tightly wound around interpretations of particles in curved space.

A Commutation relations in Rindler coordinates

I will derive $[\phi(\tilde{\eta},\xi), \partial_{\eta'}\phi(\eta',\xi')|_{\eta'=\eta}] = i\delta(\xi-\xi')$. Note that

$$\begin{aligned}\partial_{\eta'}\tilde{f}(\eta',\xi') &= \left(\frac{\partial t'}{\partial\eta'}\partial_{t'} + \frac{\partial x'}{\partial\eta'}\partial_{x'}\right) f(t',x') \\ &= e^{a\xi'}\left((\cosh a\eta')\partial_{t'} + (\sinh a\eta')\partial_{x'}\right) f(t',x').\end{aligned} \tag{22}$$

We'll start with the Schwinger function of eq.(10) in (t,x) coordinates, having performed the k^0 contour integral.

$$G\left((t,x),(t',x')\right) = -i\int \frac{dk^1}{2k^1}\left[e^{ik^1((x-x')-(t-t'))} - e^{ik^1((x-x)'+(t-t'))}\right]. \tag{23}$$

Then

$$\begin{aligned}\partial_{\eta'}G\left((t,x),(t',x')\right) &= e^{a\xi'}(\cosh a\eta' - \sinh a\eta')\int\frac{dk^1}{2}e^{ik^1((x-x')-(t-t'))} \\ &\quad + e^{a\xi'}(\cosh a\eta' + \sinh a\eta')\int\frac{dk^1}{2}e^{ik^1((x-x')+(t-t'))} \\ &= e^{a\xi'}e^{-a\eta'}\int\frac{dk^1}{2}e^{ik^1((x-x')-(t-t'))} + e^{a\xi'}e^{a\eta'}\int\frac{dk^1}{2}e^{ik^1((x-x')+(t-t'))}.\end{aligned} \tag{24}$$

Since we are interested in the commutation relations for $\eta' = \eta$, we can substitute above

$$\begin{aligned}(x-x')-(t-t') &= a^{-1}(e^{a\xi}-e^{a\xi'})\left(\cosh a\eta - \sinh a\eta\right) \\ &= a^{-1}(e^{a\xi}-e^{a\xi'})e^{-a\eta} \\ (x-x')+(t-t') &= a^{-1}(e^{a\xi}-e^{a\xi'})\left(\cosh a\eta + \sinh a\eta\right) \\ &= a^{-1}(e^{a\xi}-e^{a\xi'})e^{a\eta},\end{aligned} \tag{25}$$

giving

$$\partial_{\eta'} G\left((t,x),(t',x')\right)|_{\eta=\eta'} = e^{a\xi'} e^{-a\eta} \int \frac{dk^1}{2} e^{ik^1 a^{-1}(e^{a\xi} - e^{a\xi'})e^{-a\eta}} + e^{a\xi'} e^{a\eta} \int \frac{dk^1}{2} e^{ik^1 a^{-1}(e^{a\xi} - e^{a\xi'})e^{a\eta}}. \tag{26}$$

Now change variables. For the first integral, let $k = k^1 a^{-1} e^{-a\eta}$ and for the second integral, let $k = k^1 a^{-1} e^{a\eta}$. This leads to

$$\begin{aligned} \partial_{\eta'} G\left((t,x),(t',x')\right)|_{\eta=\eta'} &= a e^{a\xi'} \int dk e^{ik(e^{a\xi} - e^{a\xi'})} \\ &= a e^{a\xi'} \delta(e^{a\xi} - e^{a\xi'}) \\ &= \delta(\xi - \xi') \end{aligned} \tag{27}$$

Remember that $[\phi(\tilde{\eta}, \xi), \partial_{\eta'}\phi(\eta', \xi')|_{\eta'=\eta}] = i\partial_{\eta'} G\left((t,x),(t',x')\right)|_{\eta=\eta'}$ so this proves the commutation relation. The other two commutation relations also can be derived.