From representations to building blocks – the ascendancy of quarks

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August 27, 2021

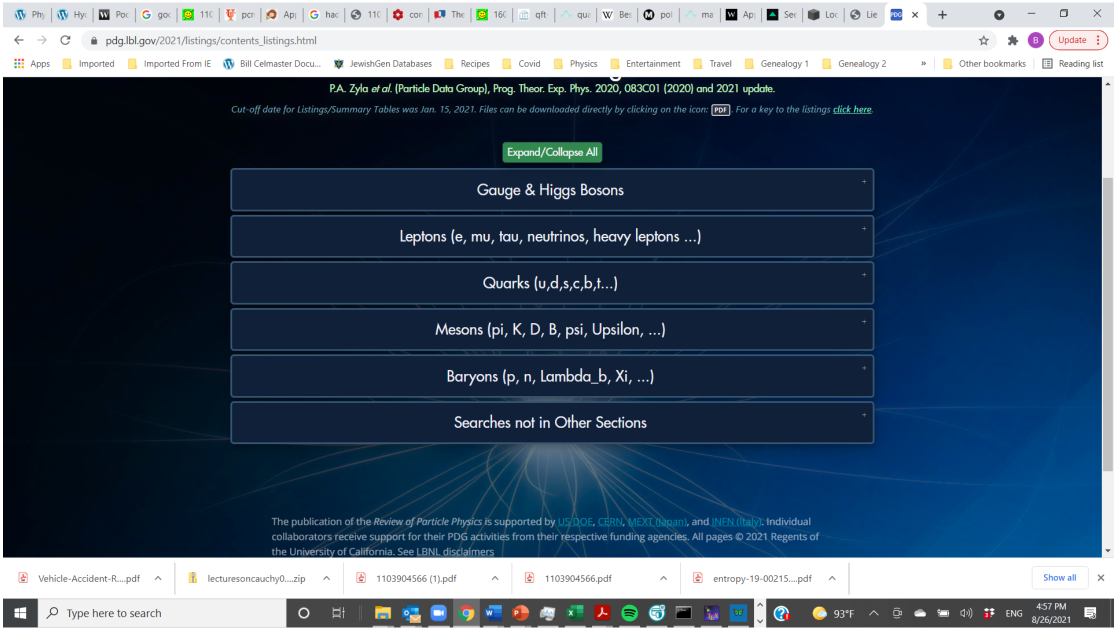
First, how to build particles out of quarks.

Second, how to relate quarks to the SU(3) symmetry from last time.

Remark on sources: I've browsed the internet for presentations on quarks and ended up taking some screenshots that appear below. Unfortunately, I was careless in noting the origin of these screenshots and no longer know their authors, much less their URL's.

1 Building particles out of quarks

The best place to see the zoo of particles is the yearly publication of the Particle Data Group. Here is a dive.


pdg.lbl.gov/2021/listings/contents_listings.html
P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2020) and 2021 update.
Cut-off date for Listings/Summary Tables was Jan. 15, 2021. Files can be downloaded directly by clicking on the icon: PDF. For a key to the listings click here.
Expand/Collapse All
Gauge & Higgs Bosons
Leptons (e, mu, tau, neutrinos, heavy leptons ...)
Quarks (u,d,s,c,b,t...)
Mesons (pi, K, D, B, psi, Upsilon, ...)
Baryons (p, n, Lambda_b, Xi, ...)
Searches not in Other Sections
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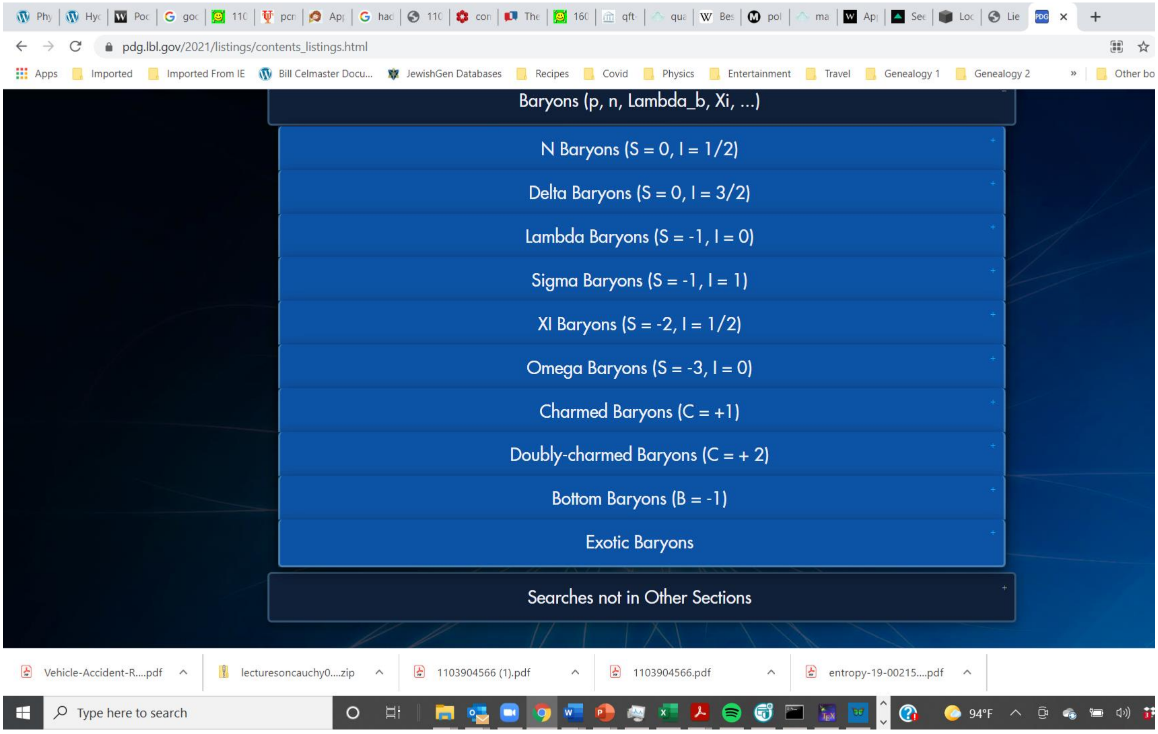
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Covid
Physics
Entertainment
Travel
Genealogy 1
Genealogy 2
Baryons (p, n, Lambda_b, Xi, ...)
N Baryons (S = 0, I = 1/2)
Delta Baryons (S = 0, I = 3/2)
Lambda Baryons (S = -1, I = 0)
Sigma Baryons (S = -1, I = 1)
XI Baryons (S = -2, I = 1/2)
Omega Baryons (S = -3, I = 0)
Charmed Baryons (C = +1)
Doubly-charmed Baryons (C = + 2)
Bottom Baryons (B = -1)
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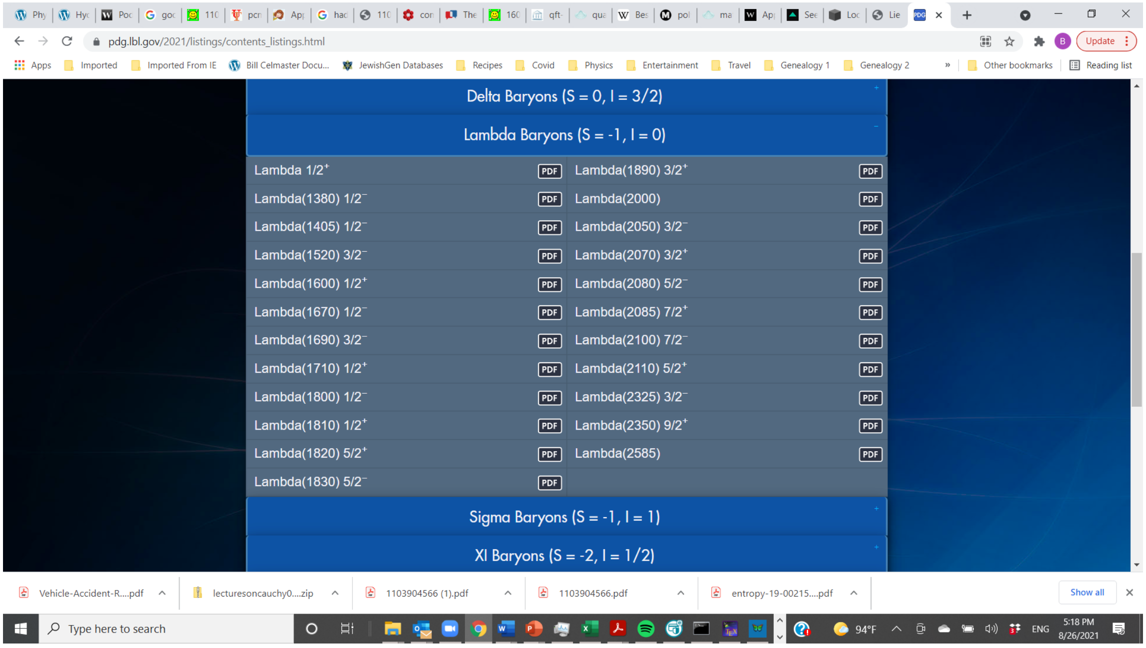

Lambda $1/2^+$ PDF	Lambda(1890) $3/2^+$ PDF
Lambda(1380) $1/2^-$ PDF	Lambda(2000) PDF
Lambda(1405) $1/2^-$ PDF	Lambda(2050) $3/2^-$ PDF
Lambda(1520) $3/2^-$ PDF	Lambda(2070) $3/2^+$ PDF
Lambda(1600) $1/2^+$ PDF	Lambda(2080) $5/2^-$ PDF
Lambda(1670) $1/2^-$ PDF	Lambda(2085) $7/2^+$ PDF
Lambda(1690) $3/2^-$ PDF	Lambda(2100) $7/2^-$ PDF
Lambda(1710) $1/2^+$ PDF	Lambda(2110) $5/2^+$ PDF
Lambda(1800) $1/2^-$ PDF	Lambda(2325) $3/2^-$ PDF
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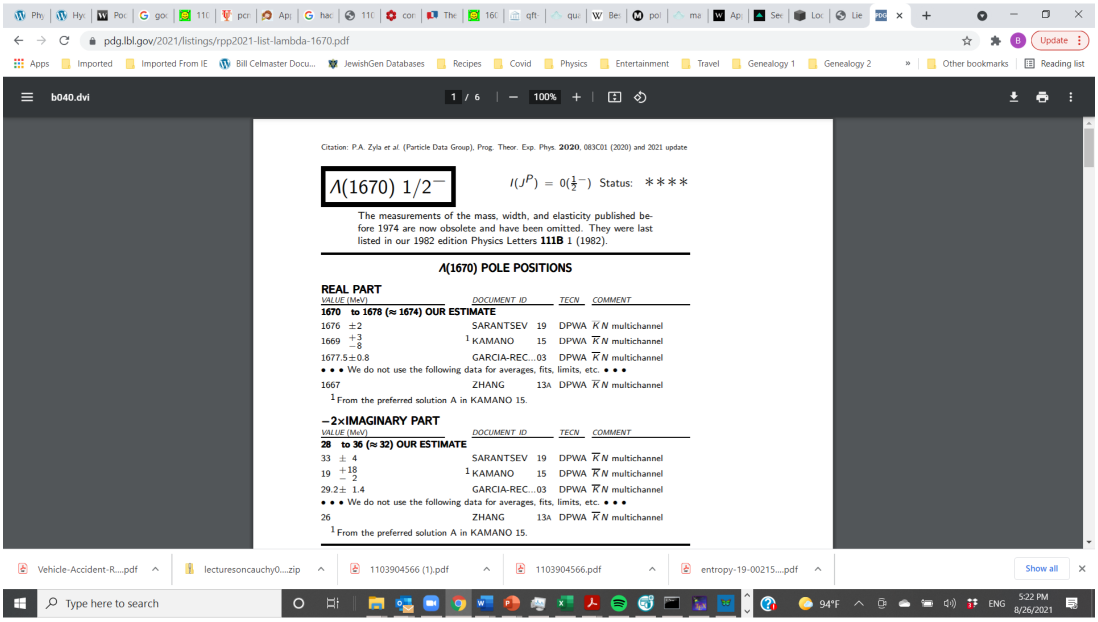
Lambda(1820) $5/2^+$ PDF	Lambda(2585) PDF
Lambda(1830) $5/2^-$ PDF	



Citation: P.A. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2020**, 083C01 (2020) and 2021 update

$\boxed{\Lambda(1670)\ 1/2^-}$ $I(J^P) = 0(\frac{1}{2}^-)$ Status: ∗∗∗∗

The measurements of the mass, width, and elasticity published before 1974 are now obsolete and have been omitted. They were last listed in our 1982 edition Physics Letters **111B** 1 (1982).

Λ(1670) POLE POSITIONS

REAL PART

VALUE (MeV)	*DOCUMENT ID*		*TECN*	*COMMENT*
1670 to 1678 (≈ 1674) OUR ESTIMATE				
1676 ±2	SARANTSEV	19	DPWA	$\overline{K}N$ multichannel
1669 $^{+3}_{-8}$	[1] KAMANO	15	DPWA	$\overline{K}N$ multichannel
1677.5±0.8	GARCIA-REC...	03	DPWA	$\overline{K}N$ multichannel
• • • We do not use the following data for averages, fits, limits, etc. • • •				
1667	ZHANG	13A	DPWA	$\overline{K}N$ multichannel

[1] From the preferred solution A in KAMANO 15.

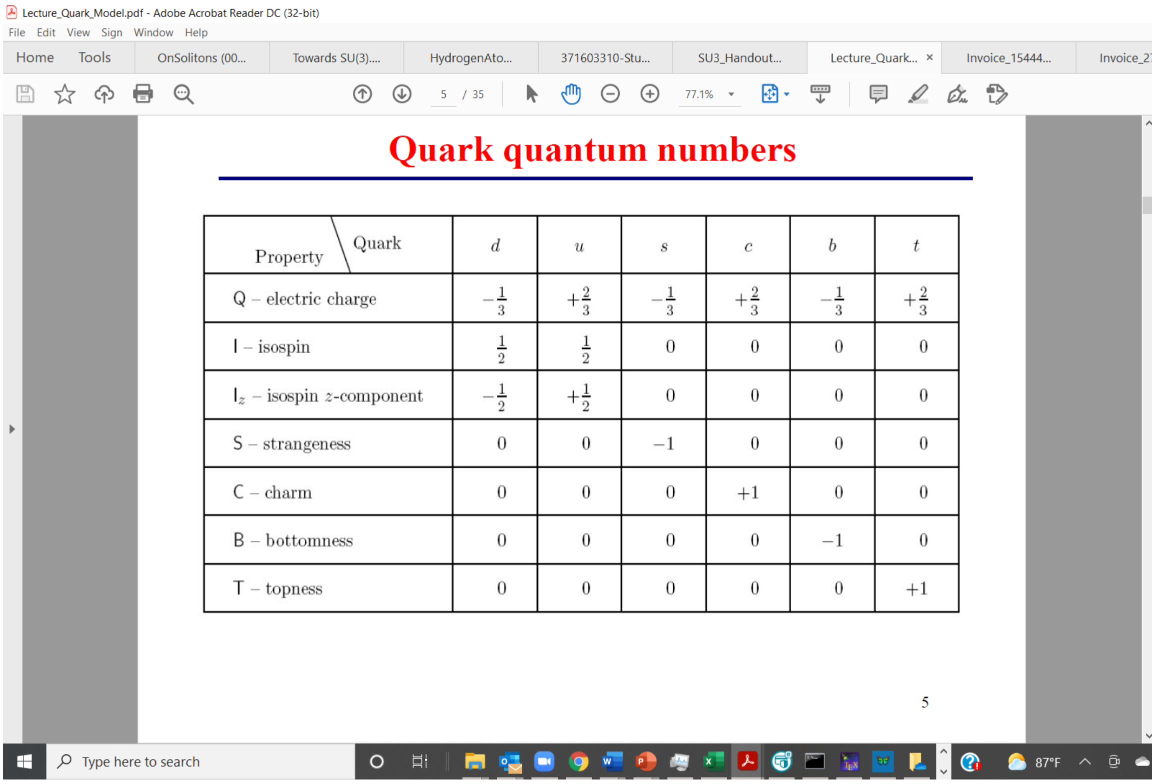
−2×IMAGINARY PART

VALUE (MeV)	*DOCUMENT ID*		*TECN*	*COMMENT*
28 to 36 (≈ 32) OUR ESTIMATE				
33 ± 4	SARANTSEV	19	DPWA	$\overline{K}N$ multichannel
19 $^{+18}_{-\ 2}$	[1] KAMANO	15	DPWA	$\overline{K}N$ multichannel
29.2± 1.4	GARCIA-REC...	03	DPWA	$\overline{K}N$ multichannel
• • • We do not use the following data for averages, fits, limits, etc. • • •				
26	ZHANG	13A	DPWA	$\overline{K}N$ multichannel

[1] From the preferred solution A in KAMANO 15.

1.1 Quark building blocks

There are 6 flavors of quark that we know about today. When quarks were first hypothesized, there were only 3 flavors – up, down and strange. Here are their properties as well as properties of the remaining quarks (charm, bottom and top).



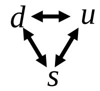
Quark quantum numbers

Property \ Quark	d	u	s	c	b	t
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$
I – isospin	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
I_z – isospin z-component	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	0
S – strangeness	0	0	-1	0	0	0
C – charm	0	0	0	$+1$	0	0
B – bottomness	0	0	0	0	-1	0
T – topness	0	0	0	0	0	$+1$

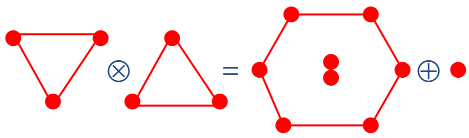
5

We combine these to produce hadrons.

Mesons: $q\bar{q}$-states



$q\bar{q}$ combinations: $3\otimes\bar{3}=1\oplus8$



Examples:

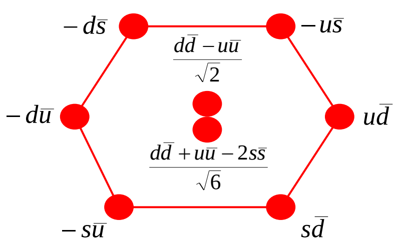
$$\boxed{\begin{array}{l} \qquad\qquad\qquad\quad K^{+}K^{0} \\ S=0:\eta,\eta' \quad \pi^{+} \quad \pi^{0} \quad \pi^{-} \\ \qquad\qquad\qquad\quad K^{-}\bar{K}^{0} \end{array}}$$

$$\boxed{\begin{array}{l} \qquad\qquad\qquad\quad K^{*+}K^{*0} \\ S=1:\omega,\varphi \quad \rho^{+} \quad \rho^{0} \quad \rho^{-} \\ \qquad\qquad\qquad\quad K^{*-}\bar{K}^{*0} \end{array}}$$

Singlet:

$$\frac{u\bar{u}+d\bar{d}+s\bar{s}}{\sqrt{3}}$$

Octet:



Meson masses

If SU(3) would be exact all particle masses within a multiplet identical

SU(3) symmetry broken by:

1. ***u-, d- and s-quark mass differences (singlet+octet→mixed "nonet")***
2. ***contributions from quark spin-spin interactions***

$$2\,\vec{s}_1\cdot\vec{s}_2 = (\vec{s}_1+\vec{s}_2)^2-(\vec{s}_1)^2-(\vec{s}_2)^2 \begin{cases} \text{S=1: } \vec{s}_1\cdot\vec{s}_2 = +1/4 \\ \text{S=0: } \vec{s}_1\cdot\vec{s}_2 = -3/4 \end{cases}$$

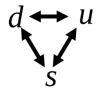
$$\text{Ansatz}: \quad M(\text{meson}) = m_1 + m_2 + A\frac{\vec{s}_1\cdot\vec{s}_2}{m_1 m_2} = \begin{cases} S=1: \quad m_1+m_2+A\dfrac{\hbar^2}{4m_1m_2} \\ S=0: \quad m_1+m_2-A\dfrac{3\hbar^2}{4m_1m_2} \end{cases}$$

Fit ⇒ m_u=m_d=310 MeV and m_s=483 MeV

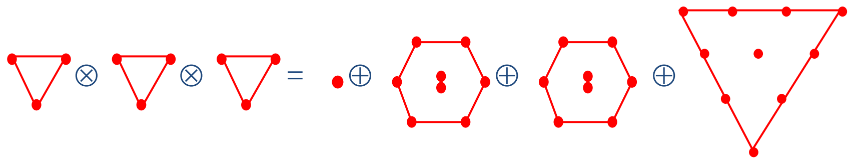
Meson nonet S=0	**fit mass**	**exp mass**
π (3)	**140**	**138**
K (2x2)	**484**	**496**
η (1)	**559**	**549**
η' (1)	**---**	**958**

Meson nonet S=1	**fit mass**	**exp mass**
ρ (3)	**780**	**776**
K (2x2)	**896**	**892**
ω (1)	**780**	**783**
ϕ (1)	**1032**	**1020**

Baryons: qqq-states

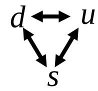


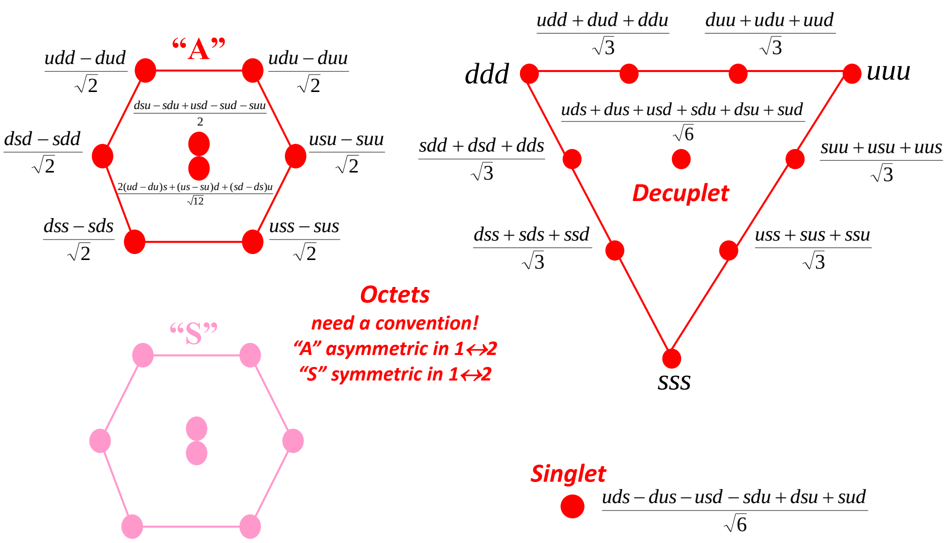
qqq combinations : $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$



Same recipe as for mesons; bit more complicated

Baryons: qqq-states





Real, physical particles were found to match the baryon decuplet properties: all have spin = 3/2 and about the same mass

BARYONS (Spin $\frac{3}{2}$)

Baryon	Quark content	Charge	Mass	Lifetime	Principal decays
Δ	uuu, uud, udd, ddd 0	$+2, +1, 0, -1$	1232	0.6×10^{-23}	$N\pi$
Σ^*	uus, uds, dds −1	$+1, 0, -1$	1385	2×10^{-23}	$\Lambda\pi, \Sigma\pi$
Ξ^*	uss, dss −2	$0, -1$	1533	7×10^{-23}	$\Xi\pi$
Ω^-	sss −3	-1	1672	0.82×10^{-10}	$\Lambda K^-, \Xi^0\pi^-, \Xi^-\pi^0$

2 Building representations of the rotation group

- In physical systems, a vector, $\mathbf{v}$ is defined to be a triplet (v_1, v_2, v_3) which transforms under the rotation $\mathbf{R}$ as

$$v_i \to \sum_j R_{ij} v_j \tag{1}$$

where R_{ij} are the elements of the 3 x 3 rotation matrix $\mathbf{R}$.

- A tensor (of rank 2) $\mathbf{T}$ is a 2-index object which transforms under the rotation $\mathbf{R}$ as

$$T_{i_1 i_2} \to \sum_{j_1, j_2} R_{i_1 j_1} R_{i_2 j_2} T_{j_1 j_2} \tag{2}$$

- Now construct a new object, the trace of $\mathbf{T}$, $S = \mathrm{tr}(\mathbf{T}) = T_{11} + T_{22} + T_{33}$. How does S transform under the rotation $\mathbf{R}$?

$$\begin{aligned} S \to & \sum_{ij} \left(R_{1i}R_{1j} + R_{2i}R_{2j} + R_{3i}R_{3j}\right) T_{ij} \\ &= \sum_{ijk} R_{ij} T_{jk} R^T_{ki} \\ &= \mathrm{Tr}(\mathbf{RTR}^T) \\ &= \mathrm{Tr}(\mathbf{T}) = S. \end{aligned} \tag{3}$$

The last line follows from the fact that rotation matrices are orthogonal, i.e., $\mathbf{R}^{-1} = \mathbf{R}^T$ and from the fact that $\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{CAB})$.

- Since S transforms into itself under rotations, it is a 1D representation of the rotations – i.e., a scalar.

- It is also easy to show that each of the symmetric tensor $S_{ij} = \frac{T_{ij}+T_{ji}}{2} - \delta_{ij}\frac{S}{3}$ and the antisymmetric tensor $A_{ij} = \frac{T_{ij}-T_{ji}}{2}$ are transformed into themselves under rotations. $\mathbf{S}$ is 5-dimensional, and $\mathbf{A}$ is 3-dimensional. Furthermore, each of S, $\mathbf{S}$ and $\mathbf{A}$ are irreducible representations, meaning that, for each of those quantities, any element can be some other element by means of some rotation (of course, since the scalar representation only has one element, irreducibility doesn't mean anything for that case). We write

$$3 \otimes 3 = 1 \oplus 3 \oplus 5 \tag{4}$$

What this means is as follows: $3 \otimes 3$ denotes the representation which transforms as a tensor product of vectors (a vector is an irreducible representation of dimension 3), specifically as eq. (2). The right hand side denotes a direct sum of (irreducible) representations of dimension 1, 3 and 5.

- An example of a tensor (of rank 2) is the product of two vectors.

$$\hat{T}_{ij} = v_i w_j. \tag{5}$$

Convince yourself that $\hat{\mathbf{T}}$ transforms as a tensor.

- For the vector product above we have

$$\begin{aligned} S &= v_1 w_1 + v_2 w_2 + v_3 w_3 \\ S_{ij} &= \frac{v_i w_j + w_i v_j}{2} - \frac{S}{3}\delta_{ij} = S_{ji} \\ A_{ij} &= \frac{v_i w_j - v_j w_i}{2} = -A_{ji} \end{aligned} \tag{6}$$

An example of how the decomposition works is this:

$$\begin{aligned} v_1 w_2 &= S_{12} + A_{12} \\ v_3 w_3 &= S_{33} + \frac{S}{3} \end{aligned} \tag{7}$$

- **Why is this useful?** Imagine a nucleus with 2 electrons, represented by wavefunctions $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$. Further imagine that each wavefunction can be written as a linear combination of the spherical harmonics from last time (write these in polar coordinates for now).

$$\begin{aligned} Y_1^{-1} &= \sqrt{\frac{3}{8\pi}} e^{-i\phi} \sin\theta \\ Y_1^0 &= \sqrt{\frac{3}{4\pi}} \cos\theta \\ Y_1^1 &= -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin\theta. \end{aligned} \tag{8}$$

Recall that these three functions transform into one another under rotations, and therefore constitute the components of a (physical) vector. Just to be concrete, write

$$\begin{aligned} \psi_1 &= v_1 Y_1^{-1} + v_2 Y_1^0 + v_3 Y_1^1 \equiv (v_1, v_2, v_3) \equiv \mathbf{v} \\ \psi_2 &= w_1 Y_1^{-1} + w_2 Y_1^0 + w_3 Y_1^1 \equiv (w_1, w_2, w_3) \equiv \mathbf{w} \end{aligned} \tag{9}$$

- We see that wave-function products of the type $\psi_1\psi_2$, can be expressed abstractly as tensor products of the form $v_i w_j$ and can therefore be decomposed into terms from the irreducible representations of dimensions 1 (S), 3 (A_{ij}) and 5 (S_{ij}).
- Since the Hamiltonian is rotationally invariant, the energy eigenvalues (continuing to assume each electron is in the 3-D representation built of the the above spherical harmonics) then have degeneracy 1 or 3 or 5.

- We can generalize the definitions of vectors and tensor products, as well as product-decompositions. For example, instead of writing S_{ij}, which has 5 independent components, we can write $\hat{v}_k$, an object where the index k ranges from 1 to 5. The action of rotations on this object would a 5 x 5 matrix that could be determined (although not pretty). Then we could form tensor products of the form $\hat{v}_k\hat{w}_l$ (25 separate indices) or $\hat{v}_k w_j$ (15 separate indices) and could rewrite these (i.e. 'decompose these') in terms of components of irreducible representations.

SUMMARY – An example: If we discovered a 5-fold degeneracy in some molecular spectrum, we would probably interpret this as coming from a rotational (irreducible) representation of dimension 5. This could be a single-electron wavefunction in its dimension-5 representation. Or it could be due to 2 electrons, each in their dimension-3 (vector) representation but in a symmetric combination (which in turn decomposes into a scalar and a dimension 5 piece). *I call these building blocks.*

3 Building representations of isospin (SU(2))

- Recall that 3 pions were discovered with approximately the same mass of about 140 MeV: π^0, π^+ and π^-. These were hypothesized to be in a 3-D irreducible representation of SU(2), also known as isospin. Notice that this SU(2) has nothing to do with rotations of coordinates but instead represents some other inherent new characteristic of particles.

- The question is "Is the pion really a single indivisible object with internal forces (kind of like a hydrogen atom) and an isospin feature which is in its 3D representation?). Or is the pion perhaps made of building blocks?"

- Hypothesize that the pions are made of 2 building blocks? How can we construct a 3D representation? Imagine each block has isospin 1/2, i.e. a 2-D representation.

$$2 \otimes 2 = 1 \oplus 3 \tag{10}$$

This equation is sometimes written in terms of the j value,[1] which can be confusing, but here it is in case you are comparing to other notes.

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{0} \oplus \mathbf{1}. \tag{11}$$

Anyway, the point is that certain combinations of a product of two isospin-$\frac{1}{2}$ states, result in a 3D (isospin 1) representation leading to 3-fold degeneracy.

- If there really are building blocks, we would expect to also see a product combination which is 1D (a scalar), hence a one-fold degeneracy. In fact, such a particle was found (with isospin 0) and is known as the η' with a mass of 958 MeV. However, if the pions and η' are made of the same two building blocks but in different combinations, why are their masses so different? That is a story for a different time and place, but was initially one of the reasons why the building-block hypothesis didn't make much headway.

- If the building-block hypothesis were reasonable, then we'd expect to see the building blocks as isospin-$\frac{1}{2}$ particles with two-fold degeneracy. In fact, the neutron and proton have isospin $\frac{1}{2}$ and have approximately the same mass of 958 MeV. Again, the question would be "why are the building blocks so much heavier than the pions?".

- Later, we came to realize that the neutron and proton were not building blocks. Instead, they come from 3 building blocks for which

$$2 \otimes 2 \otimes 2 = 2 \oplus 2 \oplus 4 \tag{12}$$

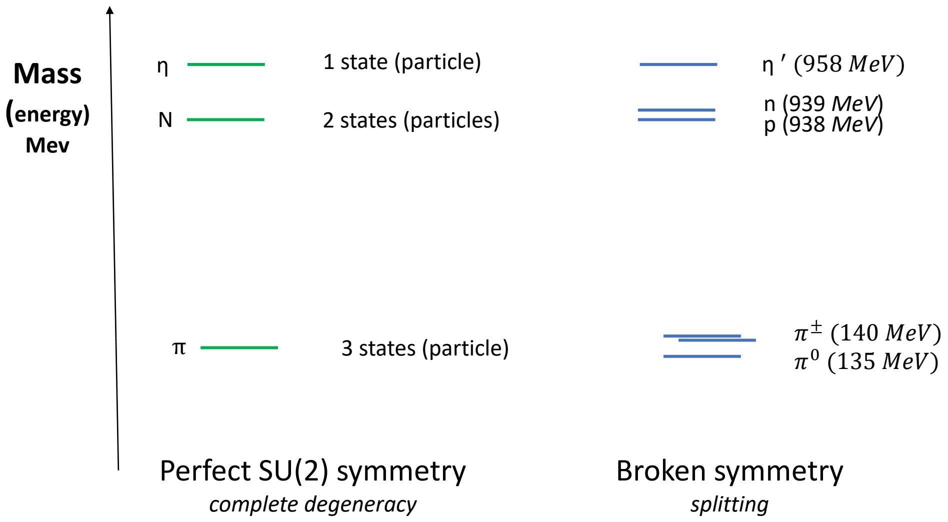
  or in the more conventional notation

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{\frac{1}{2}} \oplus \mathbf{\frac{1}{2}} \oplus \mathbf{\frac{3}{2}}. \tag{13}$$

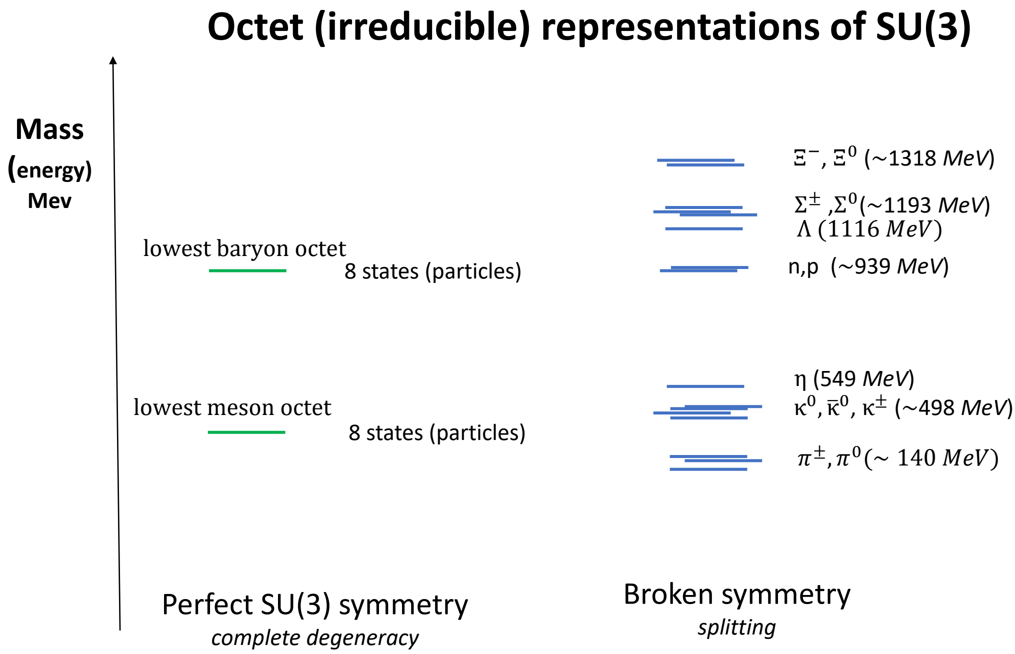
  We see that this decomposition includes 2-dimensional (isospin-$\frac{1}{2}$) representations, one of which turns out to account for the neutron and proton.

[1] Remember the convention that the dimensionality, d is $d = 2j + 1$ where j is the total angular momentum.

Irreducible representations of SU(2)



4 Building blocks for SU(3) – quarks



Physicists hypothesized that there was an overall SU(3) symmetry responsible for the structure of baryons. Some of the first sets of classifications of particles, arranged them into 8-fold degeneracy octets (sometimes called **The eightfold way**).

- We'd expect the low-dimension representations to appear, but no-one has seen the 3D representation (3-fold degeneracy).

- Building block constructions: There are two non-equivalent 3D representations, written as 3 and $\bar{3}$. Suppose these are the building blocks. **THESE ARE KNOWN RESPECTIVELY AS QUARKS and ANTIQUARKS.** The triplet components are given the names **down**, **up**, and **strange**. So if we were able to see the quarks, they would appear as three particles with the 'flavor' (identification) up, down and strange.

$$\begin{aligned} 3 \otimes 3 &= 6 \oplus \bar{3} \\ 3 \otimes \bar{3} &= 1 \oplus 8 \\ 3 \otimes 3 \otimes 3 &= 10 \oplus 8 \oplus 8 \oplus 1 \end{aligned} \tag{14}$$

We see the octet representations appearing as components of $3 \otimes \bar{3}$ and also of $3 \otimes 3 \otimes 3$.

- We then say that a neutron is a combination of 3 quarks, and a pion is a combination of a quark and anti-quark.
- Notice that both the pion and neutron also appear as particles with certain values of isospin. Isospin is a subgroup of SU(3).
- Isospin SU(2) is a less 'broken' symmetry than SU(3). That means the isospin multiplets tend to be almost degenerate in mass, whereas the SU(3) multiplets bunch together but aren't very degenerate.