What next? A review of the QFT we've done, and thoughts about what to do next

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1 Proposals for what to do next

- Consolidate and then extend what we've done in QFT by reading through a high-level undergraduate text. One such book is "Modern Particle Physics" by Mark Thomson. It is well-reviewed and appears readable. I could summarize key points as we read through them, or elaborate on certain parts of the text as desired.
- Continue tackling special topics based on customized notes. There's lots of fun areas to explore but things will become increasingly fragmented.
- Choose a different text after we decide collectively what kind of thing we want to focus on.

2 What does a Quantum Field Theorist do?

1. Model building: This is where we try to come up with new Lagrangians designed to exhibit patterns (e.g. symmetries) observed in experiments, or designed to avoid theoretical inconsistencies etc. We've done some model building by looking for Lagrangians that are invariant under representations of the Lorentz group, and that led us for example, to the Dirac equation.
2. Compare models to experiments: This is where we use computational tools such as perturbation theory (e.g. Feynman diagrams), lattice techniques, group representations etc. to predict particle masses, properties and scattering amplitudes. We've done a bit of that in the past month in analyzing particle masses and properties.

3. Analyze theoretical properties of the field theories associated with various kinds of Lagrangians. For example, one can look for proofs that in each order of perturbation theory, all the relevant integrals are convergent. Or one can look for mathematical equivalences between superficially different Lagrangian theories. We did this kind of thing as we derived, using the path integral, terms of the perturbation expansion.

In any reasonable one-year introduction to QFT, all three of the above are covered to some extent. Mark Thomson's text starts off with a bit of extra emphasis on point 2 (experimental predictions) whereas Kachelriess starts off heavily oriented towards point 3 (the theory). Lancaster, after reviewing some quantum mechanics and harmonic oscillators, starts off with point 1 (model building).

3 High Level Review

- The fundamental objects of QFT are fields – indexed functions of space and time $\phi^i(t,x)$.
- The theory of physics described by QFT is specified by a Lagrangian – a function of fields and their space-time derivatives, $\mathcal{L}(\phi^i, \partial_\mu\phi^i, \partial_\mu\partial_\nu\phi^i, ...)$
- We have explored two equivalent formulations of the theory based on a given Lagrangian.
  - The Canonical Formulation *Text – Lancaster and Blundell*: This approach is used primarily for identifying the fundamental particles of the theory and for constructing relativistic generalizations of the Hamiltonian and the Schrodinger equation. For example, we obtained the Dirac equation, and we also examined properties of particle mass-degeneracies.
    * First treat the fields as complex functions, and write down the Euler-Lagrange equations.

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} = \frac{\partial \mathcal{L}}{\partial \phi^i} \tag{1}$$

      The Hamiltonian, H, is defined to be a particular combination of the fields and their derivatives.

* Then assert that the fields are really complex linear operators (on a Hilbert Space) which obey those Euler-Lagrange equations. *At this point we have only asserted that the fields are operators but we haven't said what the Hilbert space is.*
* We learn more about the operators and their Hilbert spaces by stating the **canonical commutation relations** of the operators, e.g.

$$[\phi^i(t,\mathbf{x}), \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^j)}(t,\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}')\delta_{ij} \tag{2}$$

* Generally we solve these equations by perturbation theory. We rewrite the Lagrangian in terms of a free part $\mathcal{L}_0$ and an interacting part $\lambda\mathcal{L}_I$ as

$$\mathcal{L} = \mathcal{L}_0 + \lambda\mathcal{L}_I \tag{3}$$

The free part consists of terms that are quadratic in the fields, and the interacting part consists of the remainder. The parameter λ is called *the coupling constant* and is a real number. When it is small, we can derive a perturbative expansion in orders of λ.

* The Euler-Lagrange equation for the free part can be solved exactly. Then, using the canonical commutation relations, we can determine the Hilbert space and therefore the fundamental particles of the theory. There are two important concepts we've encountered.
    · The lowest-energy eigenstate is called **the vacuum state**, or simply **the vacuum**.
    · Single particles of momentum $\mathbf{k}$ are 'created' by applying the operator $a_k^{i\dagger}$ to the vacuum state. That operator is called the **creation operator** and its adjoint is called the **annihilation operator**. Both the annihilation and creation operators can be constructed as linear combinations of the fields.

- The Feynman Path Integral *Text – Kachelriess*: This approach is used primarily for computing scattering probabilities, where two particles collide (*scatter*) and probabilities are computed for the outcomes (particle such-and-such with momentum such-and-such). Since the path integral approach involves only complex-valued fields and not operators, it can be much easier to use in certain circumstances, than the canonical formalism. In addition, I personally find the path integral much easier to visualize than the operator approach. In fact, since it's an integral which sums up the phases corresponding to each path, one can approximate it by numerical computation and this technique (lattice quantum theory) is very successful for computing quantities that aren't amenable to perturbation theory.

- First use the above Lagrangian to construct the **generating functional** $Z[J]$

$$Z[J] \equiv \int \mathcal{D}\phi^i e^{i(\int d^4x \mathcal{L}(\phi^i)(x) + \sum_i J^i(x)\phi^i(x) + i\epsilon)} \tag{4}$$

  where $\mathcal{D}\phi^i$ denotes a multi-dimensional integral with one integral for each index i and for each point $(t, \mathbf{x})$ of space-time, and ϕ^i and J^i are complex-valued functions.

- Next define the Green functions $G^{i_1 i_2 \ldots i_n}(x_1, x_2, ..., x_n)$ as

$$G^{i_1 \ldots i_n}(x_1, ..., x_n) = (-i)^n \frac{1}{Z[0]} \frac{\delta^n Z[J]}{\delta J^{i_1}(x_1) \ldots \delta J^{i_n}(x_n)} |_{J^i(x)=0} \tag{5}$$

- We use the Green functions to compute the scattering matrix. For example $\langle p_1^{out}, p_2^{out} | S | p_1^{in}, p_2^{in} \rangle$ denotes the probability amplitude that two particles with ingoing momenta p_1^{in}, p_2^{in} collide and end up as two particles with outgoing momenta p_1^{out}, p_2^{out}. For this example assume that the initial and final particles have mass m and are obtained from the quantum field $\phi(x)$. Then the calculation to be performed is this:

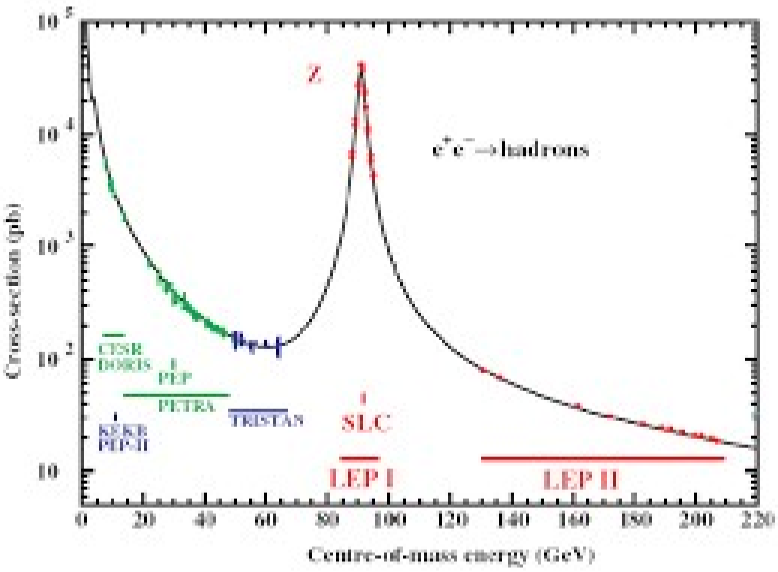
$$\begin{aligned} &\langle p_1^{out}, p_2^{out} | S | p_1^{in}, p_2^{in} \rangle = \\ &\quad (-(p_1^{out})^2 + m^2)(-(p_2^{out})^2 + m^2)(-(p_1^{in})^2 + m^2)(-(p_2^{in})^2 + m^2) \\ &\quad [i \int d^4x_1 e^{-ip_1^{in} \cdot x_1}][i \int d^4x_2 e^{-ip_2^{in} \cdot x_2}][i \int d^4x_3 e^{+ip_1^{out} \cdot x_3}][i \int d^4x_4 e^{+ip_2^{out} \cdot x_4}] \\ &\quad G(x_1, x_2, x_3, x_4). \end{aligned} \tag{6}$$

  This equality relating the S-matrix to the Green function, is an example of the LSZ theorem and is easily generalized to more ingoing and outgoing particles with multiple masses and associated with other quantum fields.

– We can also learn about the particle spectrum from Green functions and scattering experiments. For example, suppose we collide an electron and a positron. If the energy of collision is larger than about 91 GeV, then this would be enough energy to create a Z particle. It turns out that the Green function can be computed to leading order of the coupling constant and leads to a scattering matrix

$$|\langle p_1^{out}, p_2^{out}|S|p_1^{in}, p_2^{in}\rangle|^2 \propto \frac{E^2}{[(4E^2 - M_Z^2)]^2} g(\theta, ...)$$

where the centre-of-mass energy is $2E$ and the angle of scattering is θ. Indeed, in an actual experiment we see the scattering probability peaks at around 91 GeV. This illustrates that we can extract particle masses from Green functions and not only from the canonical formalism above.



4 Some interesting topics we haven't discussed (much)

* Color SU(3) – each flavor (e.g. 'strange') of quark come in 3 colors.
* Gauge/local symmetries – these lead to symmetry-derived forces such as the strong force, and to their mediating fields such as the gluon field.

* SU(3) x SU(2) x U(1) – a gauge symmetry currently regarded as the best fit to the world (part of the 'standard model').
* Broken symmetry and the Higgs meson – Susskind gave a talk on this, but there are much more straightforward treatments. Broken symmetries are responsible for particle masses.
* Asymptotic freedom – the strong interactions, when acting at short distances (high energies) become weaker and thus amenable to perturbation theory (at high energy).
* SU(5), O(10) and other generalizations of SU(3) x SU(2) x U(1) – sometimes known as Grand Unification or GUTS.
* Massive neutrinos – mostly interesting because back in the 70's, neutrinos were *known* to be massless. There were interesting astrophysical and theoretical reasons to question this, and experiments now show that neutrinos have nonzero mass.