Quark exercises and their solutions

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Some recent experiments claim to have produced tetra-quark particles – mesons built out of 4 quarks. One of the latest results was submitted for publication in March 2021 by a collaboration from the Large Hadron Collider at CERN. The authors speculate that this particle has composition "$\bar{c}cu\bar{s}$". This notation is read as "anti-charm-quark, charm-quark, up-quark, anti-strange-quark".

Recall articles have the same mass but opposite quantum numbers (e.g. strangeness, charge, etc) from their anti-particles

Recall that the strange quark has a strangeness of -1 and a charge of -1/3. The up quark has a strangeness of 0 and a charge of +2/3.

PROBLEM 1: What is the expected strangeness and approximate mass of this newly discovered tetra-quark meson?

Assumptions

- The Hamiltonian of the tetra-quark system is

$$H = m_u + m_s + m_c + m_c + V(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}) + A\frac{\sum s_i \cdot s_j}{m_i m_j} \qquad (1)$$

  where V is some potential energy dependent on the 4 positions of the quarks (and probably linear in the distance between quarks) and A is a spin-coupling coefficient.[1]

[1] The form of the Hamiltonian is actually far more complicated but for the purposes of this exercise, assume that all potential energy and spin interactions contribute negligible energy to the lowest-energy eigenfunctions.

- The most significant energy contribution of the Hamiltonian comes from the mass-energies of the quarks so you can ignore contributions of the potential V and the term proportional to A. [2]
- Assume, based on fitting masses of more conventional particles, that $m_u = 300\text{MeV}$, $m_s = 500\text{MeV}$, $m_c = 1500\text{MeV}$.

SOLUTION TO PROBLEM 1

- The 'strangeness' content of the tetra-quark is determined by the strangeness values of its constituent quarks. Only the $\bar{\text{s}}$ quark (anti-strange quark) has a strangeness value. Since a strange quark has value -1, then the anti-strange quark – and hence the tetra-quark particle – has strangeness 1.
- The mass of the tetra-quark system is defined to be its ground-state energy at rest – by which I mean the lowest-energy eigenvalue of the Hamiltonian. (When I specified the Hamiltonian in eq. (1) I assumed it was at rest, otherwise there would have been an additional kinetic energy term.) By assumption for this lowest-energy state, the dominant contribution to the Hamiltonian comes from the mass terms, $m_u + m_s + m_c + m_c = (300 + 500 + 1500 + 1500)$ MeV $= 3800$ MeV. Here is the paper that discusses the discovery of this particle: `https://arxiv.org/pdf/2103.01803.pdf`. What the paper tells us is that "The most significant state, $\text{Z}_{\text{cs}}(4000)^+$, has a mass of 4003 ± 6... MeV." It's interesting to notice how close this mass is to the one we predicted.

PROBLEM 2 – HARDER: The new particle turns out to have spin 1. What other possible spins could it have had?

Hints

[2]In the future, excited states – particles with higher masses – might be found but assume the first ones found are the lightest.

- Recall that two quarks, each of spin 1/2, can combine to form degenerate-mass states of spin 0 and spin 1. In group language, we write

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \tag{2}$$

  where $\frac{1}{2}$ denotes the 2-dimensional (spin 1/2) irreducible representation, 0 denotes the 1-dimensional (spin 0) irreducible representation, and 1 denotes the 3-dimensional (spin 1, also known as vector) irreducible representation.

- Tensor product decompositions follow the associativity and distribution principles – in particular

$$(A \otimes B \otimes C \otimes D) = (A \otimes B) \otimes (C \otimes D) \tag{3}$$

  and

$$(A \oplus B) \otimes (C \oplus D) = (A \otimes C) \oplus (A \times D) \oplus (B \otimes C) \oplus (B \otimes D) \tag{4}$$

- A tensor of representation N, when tensor-producted with a scalar, is a tensor of representation N. Symbolically,

$$A \otimes 0 = A \tag{5}$$

- Reminder of tensor product of vectors.

$$1 \otimes 1 = 0 \oplus 1 \oplus 2 \tag{6}$$

  where, as mentioned above, spin 1 has dimension 3 and spin 2 has dimension 5.

SOLUTION TO PROBLEM 2

- There's nothing obvious or easy about knowing which are the irreducible representations of a Lie group. That's a subject unto itself. However, for this problem, I've given you what information you need for the solution. For those of you curious about such things, there is one irreducible representation of SO(3) (the rotation group) with each odd dimension. So there is a one-dimension, three-dimensional, five-dimensional etc. irreducible representation and these are known respectively as spin 0, spin 1, spin 2 etc. In notes, I've shown how to obtain the 5-dimensional representation as consisting of the traceless symmetric 2-tensors. I haven't shown anything beyond that, and it isn't especially obvious (although unsurprisingly, it turns out one can use generalized tensors, and then use various combinations of symmetrization and anti-symmetrization). As for SU(2), it has the same irreducible representations as SO(3) plus an irreducible representation for each even dimension. The two-dimensional and four-dimensional representations are known respectively as spin 1/2 and spin 3/2, and this pattern continues for higher dimensions.

- Since we are dealing with quarks – which are fermions – the appropriate group to consider is SU(2) rather than SO(3). That's because rotations transform fermions 'up to a phase' – in particular, if you apply a 360-degree rotation to a fermion state $|\psi\rangle$ it becomes $-|\psi\rangle$ (the minus sign would be absent for a representation of SO(3)).

- **Caveat emptor** In my hints and generally in my discussion of spin-1/2 particles, I've treated them as 2-dimensional (2 indices) objects and designated these as being in the '1/2' representation. **Strictly speaking, all of this is wrong, but I'll continue to do that much of the time!** It turns out that in most cases, the spin-1/2 particles actually are 4-dimensional and are represented by a direct sum of 2-dimensional representations. In other words, they are reducible, NOT irreducible representations. We should write the representations as $\frac{1}{2} \oplus \frac{1}{2}$. After so much emphasis on irreducible representations, it's a bit of a disappointment that our favorite particles are all in reducible representations. Rather than examining this issue (which has bearing on both mass and parity) any further, I'll go back to the pretense that quarks have 2-D representations. It turns out that this won't change the answers to questions like "what are the possible spins of composite particles?"

- The question being asked in the problem is this: if we have a system consisting of 4 quarks, each of which transform a certain way under rotations (e.g. a change of axes), then how does the overall system transform under rotations? Remember that the overall system can assemble itself into various combinations having different energies (masses). Furthermore we identify multi-quark 'particles' as collections of equal-energy (mass) bound states that transform into one another under irreducible representations of rotations. So putting these thoughts together, we see that the question of interest is ”what are the irreducible representations which can be created from combinations of 4 quarks?”

- The mathematical rendition of the above situation goes like this: Each quark q_i is a 2-index object (but see the above caveat emptor) which transforms under a rotation like $q_i \to \sum_j S_{ij} q_j$. Then if we have 4 quarks, q^1, q^2, q^3, and q^4, these transform collectively as $q^1_{i_1} q^2_{i_2} q^3_{i_3} q^4_{i_4} \to \sum_{j_1 j_2 j_3 j_4} S_{i_1 j_1} S_{i_2 j_2} S_{i_3 j_3} S_{i_4 j_4} q^1_{j_1} q^2_{j_2} q^3_{j_3} q^4_{j_4}$. This is a tensor-product representation. Schematically, we describe this transformation rule as $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$. This then gives the transformation rule for the composite particle that consists of 4 quarks. We have, in fact, constructed a 16-dimensional (2 x 2 x 2 x 2 indices) representation of SU(2). This representation is reducible. What that means is that we can decompose this 16-dimensional space into smaller-dimensional spaces which transform into themselves under the above tensor-product. Each such space describes a set of 4-quark states, whose spin corresponds to the dimensionality of that space (so for example, if one of the spaces is 3-dimensional, then those 4-quark states are spin 1).

- Now, we're ready to apply the hints and solve the problem.
  - From eq. (3) we have $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (\frac{1}{2} \otimes \frac{1}{2}) \otimes (\frac{1}{2} \otimes \frac{1}{2})$.
  - Then applying eq. (2) to each of the parentheses on the RHS, we have
    $$(\frac{1}{2} \otimes \frac{1}{2}) \otimes (\frac{1}{2} \otimes \frac{1}{2}) = (0 \oplus 1) \otimes (0 \oplus 1). \tag{7}$$
    Using the distribution principle, eq. 4 the RHS becomes
    $$(0 \oplus 1) \otimes (0 \oplus 1) = (0 \otimes 0) \oplus (0 \otimes 1) \oplus (1 \otimes 0) \oplus (1 \otimes 1). \tag{8}$$
    Now apply our rule for tensor products with scalars (eq. (5)). The RHS becomes $0 \oplus 1 \oplus 1 \oplus (1 \otimes 1)$ Finally apply the rule for tensor products of spin-1(6) to obtain $0 \oplus 1 \oplus 1 \oplus 0 \oplus 1 \oplus 2$.
  - We see that the possible spins for the tetra-quark are 0, 1 and 2. In particular, this is a meson since it has integer spin. There are three different combinations leading to a scalar particle, two combinations leading to a vector (spin-1) particle and one combination leading to spin-2.