Plane Wave in Quantum Mechanics

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I. Plane Wave

First consider the wave function of a free particle in one dimension,

$$\psi(x,t) = N \exp i(px - Et)/\hbar, \quad -\infty < x < \infty, \tag{1}$$

where N, E, and p are the amplitude, energy, and momentum of the particle, respectively. The angular frequency ω and magnitude of wave vector k are related to energy and momentum by

$$E = \hbar\omega, \quad p = \hbar k. \tag{2}$$

Since $E = p^2/2m$,

$$\omega = \frac{\hbar k^2}{2m}. \tag{3}$$

Then (1) can be written as

$$\psi(x,t) = N \exp i(kx - \omega t), \quad -\infty < x < \infty. \tag{4}$$

Question: How do we determine N?

Method 1 Suppose we assume that the free particle is confined to an one-dimensional box, [0, L]. Then $\psi(x,0)$ is zero outside the box and

$$\psi(x = 0, 0) = 0, \quad \psi(x = L, 0) = 0. \tag{5}$$

If the boundary condition (5) is imposed on $\psi(x,0)$, then the wave function becomes a standing wave and energy spectrum becomes discrete, with $N = \sqrt{\frac{2}{L}}$ [3]

$$\psi_n(x,0) = \sqrt{\frac{2}{L}} \sin(\frac{n\pi}{L}x), \quad n = 1, 2, ... \quad 0 \le x \le L, \tag{6}$$

$$E_n = \frac{\hbar^2\pi^2}{2mL^2} n^2, \quad n = 1, 2, ... \tag{7}$$

Equations (6) and (7) can be easily extended to three dimensions. *Remark 1*: Since the wavefunction is not a plane wave, a box-like boundary condition is not suitable for a plane wave.

Method 2 For normalization, consider the space dependence of the wave function only

$$\psi(k, x, 0) = N \exp ikx. \tag{8}$$

The continuous eigenvalue of the momentum operator is $\hbar k$. When a continuous spectrum is present, the standard orthonormality relation used for wave functions is [2], [4]

$$\int_{-\infty}^{\infty} dx \;\; \psi^*(k', x, 0)\psi(k, x, 0) = \delta(k - k'). \tag{9}$$

Using (8) in (9), we get

$$|N|^2 \int_{-\infty}^{\infty} dx \;\; \exp i(k - k')x = \delta(k - k'). \tag{10}$$

A useful integral representation of the Dirac delta function is [1]

$$\delta(k-k') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \;\exp i(k-k')x. \tag{11}$$

Comparing (10) and (11), for the one dimensional case we can choose

$$N = \frac{1}{\sqrt{2\pi}} \tag{12}$$

and for the three-dimensional case

$$N = \frac{1}{(2\pi)^{3/2}}. \tag{13}$$

Hence a plane wave is represented by

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\exp i(kx-\omega t), \quad -\infty < x < \infty, \tag{14}$$

$$\psi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\exp i(\mathbf{k}\cdot\mathbf{x}-\omega t), \quad -\infty < \mathbf{x} < \infty, \tag{15}$$

$$\psi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\exp i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar, \quad -\infty < \mathbf{x} < \infty. \tag{16}$$

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