Solutions to Exercises for Thomson Chapter 2.1 – 2.3.2

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The following are useful: $c = 2.9979$ x 10^8 m/s and $\hbar = 1.0546 \times 10^{-34}$ J·s = 1.0546×10^{-34} kg m^{2}s^{-1}

1. A proton has a mass of approximately 1 GeV. Convert that to MKS (S.I.) units.

   SOLUTION: A 'GeV' is a unit of energy, with a conversion of 1 GeV = 1.6021773 x $10^{-10} J$ = 1.6021773 x 10^{-10} kg m^{2}s^{-2}.

   The following argument follows the general approach for conversions involving natural units.

   - The idea of using natural units, is that you express quantities as multiples of some product of c's and $\hbar$'s, and then proceed to set $c = 1$ and $\hbar = 1$. You end up with a quantity that appears to have 'wrong' units (for example, its seems wrong to express a mass as 'GeV' or 'Joules'). But that's because you've been lazy and 'forgotten' to include the c's and $\hbar$'s (because you set them to 1). So, to get the right units, you need to first figure out what you *should* have written. Then you perform the conversion.
   - We want the mass of the proton in terms of kilograms and not Joules. So we need to use combinations of c and $\hbar$ that can be used to transform units of the form kg m^{2}s^{-2} to 'kg'.
   - The trick is to find what product has the units s^2 m^{-2}.
   - It's easy to see that the term $1/c^2$ has those units. Furthermore, that's the only combination which has those units.
   - That tells us that we *should* have said "the proton mass is approximately 1GeV/c^2 ".

- Now we can do the conversion.

$$
\begin{aligned}
m_p &= \frac{1.6021773 \text{ x } 10^{-10} \text{ kg m}^2\text{s}^{-2}}{c^2} \\
&= \frac{1.6021773 \text{ x } 10^{-10} \text{ kg m}^2\text{s}^{-2}}{(2.9979 \text{ x } 10^8 \text{ m/s})^2} \\
&= 1.78269 \text{ x } 10^{-27}\text{kg}
\end{aligned} \tag{1}
$$

2. An electron has a speed of 0.5 in natural units. What is its speed in km/sec?

   SOLUTION: Following the logic above, we've been given the speed 0.5, without units. That's wrong. The units need to be in m/s. The only combination of c and $\hbar$ which ends up as m/s, is c. So we can rewrite the speed of the electron as $v_e = 0.5c$. Then we convert. $v_e = 0.5 \text{ x } 2.9979 \text{ x } 10^8$ m/s $= 1.49895 \text{ x } 10^8$ m/s.

3. ** Start with the one-particle relationships E $= \gamma m$ and $\mathbf{p} = \gamma m \mathbf{v}$. Prove that $p^\mu p_\mu = m^2$.

   SOLUTION:

$$p^\mu = (\gamma m, \gamma m \mathbf{v}) \tag{2}$$

   so

$$p_\mu = (\gamma m, -\gamma m \mathbf{v}) \tag{3}$$

   Using the Einstein summation convention, we have $p^\mu p_\mu \equiv \sum_{\mu=0}^{\mu=3} p^\mu p_\mu$, so

$$
\begin{aligned}
p^\mu p_\mu &= (\gamma m)^2 - (\gamma m)^2 v^2 \\
&= m^2 \gamma^2 (1 - v^2) \\
&= m^2 \frac{1}{1 - v^2} (1 - v^2) = m^2
\end{aligned} \tag{4}
$$

   where we notice that the dot product of $\mathbf{v}$ with itself is v^2, and where we have also used the definition of γ.

4. Suppose a moving proton has an energy of 2 GeV. What is its speed?

   SOLUTION: Recall that $E = \gamma mc^2$. I've explicitly included the factors of c so that we don't have to go through the deduction process of figuring these out based on units. Also keeping factors of c, we write $\gamma = \left(\sqrt{1 - v^2/c^2}\right)^{-1}$.

   In Exercise 1, we learned that a proton has a mass of about 1 GeV, by which we meant that $mc^2 = 1\text{GeV}$. Since the moving proton has an energy of 2 GeV, we then see that

$$2 \text{ GeV} = \gamma(mc^2) = \gamma(1 \text{ GeV}) \tag{5}$$

so $\gamma = 2$. Then $\gamma^2 = 4$ from which we conclude that

$$\frac{1}{1 - \left(\frac{v}{c}\right)^2} = 4 \tag{6}$$

from which we obtain

$$\left(\frac{v}{c}\right)^2 = \frac{3}{4}. \tag{7}$$

Thus $v = \sqrt{3/4}c = 0.866 \text{ x } 3.00 \text{ x } 10^8$ m/s $= 2.60 \text{ x } 10^8$ m/s.

5. ** Suppose we have a system of two protons colliding head-on so that their total 3-momentum is 0 (also known as the center-of-mass frame). If each has a speed of 0.5 in natural units, what is the invariant mass of the system?

   SOLUTION: The invariant mass of the system is defined on page 37 of the text, as

   $$M_I^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \tag{8}$$

   Because the system is in its center-of-mass, the momenta are equal and opposite so $(\mathbf{p}_1 + \mathbf{p}_2) = 0$. Thus $M_I^2 = (E_1 + E_2)^2$.

   As in the previous exercises, $E_1 = \gamma mc^2 = \gamma$ x 1 GeV.

   $$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1 - 0.5^2}} = 1.155 \tag{9}$$

   Putting all this together, the invariant mass is

   $$M_I = (1.155 + 1.155) \text{ GeV} = 2.31 \text{ GeV}. \tag{10}$$

6. ** **My original problem statement was very poorly posed to the point of probably being unintelligible. So what follows is a restatement and solution.** In my notes, I showed that in the s-channel Feynman diagram (first diagram of Figure 2.2 in the text), the value of q is $(p_1 + p_2)$ so that q^2 is just the Mandelstam variable s. On page 39 of the text, the Mandelstam variable s is also said to equal $(p_3 + p_4)^2$. Show this by using the same Feynman diagram, but focusing on the right vertex. Also, notice that if momentum is

conserved at each vertex, then it is ultimately conserved between the incoming and outgoing particles.

SOLUTION: On the right vertex of that Feynman diagram, we see that conservation of momentum implies that $q = p_3 + p_4$ therefore $q^2 = (p_3 + p_4)^2$. But from the left vertex we found that $q^2 = s$. So this shows that $s = (p_3 + p_4)^2$.

Another approach which doesn't rely on Feynman diagrams, is to simply notice that the momentum of ingoing particles is the same as for outgoing particles (i.e., overall conservation of momentum for the process).

7. Problem 2.12 in the book. Prove that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

SOLUTION: Before showing the proof, it's worth mentioning that this identity is frequently used for simplifying various scattering amplitudes.

The proof that follows was provided by Steve Rubin and is much simplified from the proof that I originally offered in the first version of these solutions.

On page 39 of Thomson, the Mandelstam variables are defined as follows:

$$\begin{array}{rlcl} & \quad\textbf{A} & & \quad\textbf{B} \\ s & = (p_1 + p_2)^2 & = & (p_3 + p_4)^2 \\ t & = (p_1 - p_3)^2 & = & (p_2 - p_4)^2 \\ u & = (p_1 - p_4)^2 & = & (p_2 - p_3)^2 \end{array}$$

Start by taking the definition of all three variable from column **A**. Then, the proof proceeds as follows:

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1p_2 - 2p_1p_3 - 2p_1p_4 \\ &= p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1^2 + 2p_1p_2 - 2p_1p_3 - 2p_1p_4 \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1\left[p_1 + p_2 - (p_3 + p_4)\right] \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1\left[0\right] \quad \text{(conservation of momentum)} \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 \end{aligned}$$

8. ** Following the proof in the book for 3D, prove the continuity equation for a one-dimensional system with coordinate x. Namely,

$$\partial_x j + \partial_t \rho = 0 \tag{11}$$

where

$$j = \frac{1}{2im}\left(\psi^* \partial_x \psi - \psi \partial_x \psi^*\right). \tag{12}$$

The 1-D Schrodinger equation to use is given at the bottom of page 41 and is

$$i\frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t). \tag{13}$$

SOLUTION: Take the complex conjugate of the Schrodinger equation:

$$-i\frac{\partial \psi^*(x,t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2 \psi^*(x,t)}{\partial x^2} + V(x)\psi^*(x,t). \tag{14}$$

Multiply eq. (13) by ψ^*, multiply eq. (14) by ψ and then subtract the second product from the first. We obtain

$$\begin{aligned} -\frac{1}{2m}\left(\psi^*(x,t)\frac{\partial^2 \psi(x,t)}{\partial x^2} - \psi^(x,t)\frac{\partial^2 \psi * (x,t)}{\partial x^2}\right) \\ + \left(V(x)\psi^*(x,t)\psi(x,t) - V(x)\psi(x,t)\psi^*(x,t)\right) = \\ i\left(\psi^*(x,t)\frac{\partial \psi(x,t)}{\partial t} + \psi(x,t)\frac{\partial \psi^*(x,t)}{\partial t}\right). \end{aligned} \tag{15}$$

The terms that include the potential V can be seen to cancel each other. The above equation can then be shown (perform the derivatives to convince yourself) to be equivalent to

$$\begin{aligned} -\frac{1}{2m}\frac{\partial}{\partial x}\left(\psi^*(x,t)\frac{\partial \psi(x,t)}{\partial x} - \psi(x,t)\frac{\partial \psi^*(x,t)}{\partial x}\right) \\ = i\frac{\partial}{\partial t}\left(\psi^*(x,t)\psi(x,t)\right). \end{aligned} \tag{16}$$

If we then set $j = \frac{1}{2im}\left(\psi^* \partial_x \psi - \psi \partial_x \psi^*\right)$ and $\rho = \psi^*(x,t)\psi(x,t)$, then above equation becomes

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0, \tag{17}$$

which is what we set out to show.