Thomson Chapter 2.1 – 2.3.2 Underlying Concepts Part I

Bill Celmaster

November 3, 2021

1 Exercises

The four exercises with stars are probably the most instructive. I'll only provide answers is someone asks for them. Also consider the text problems 2.1, 2.2, 2.4, and 2.5.

1. A proton has a mass of approximately 1 GeV. Convert that to MKS (S.I.) units.

2. An electron has a speed of 0.5 in natural units. What is its speed in km/sec?

3. ** Start with the one-particle relationships $\mathrm{E} = \gamma m$ and $\mathbf{p} = \gamma m \mathbf{v}$. Prove that $p^\mu p_\mu = m^2$.

4. Suppose a moving proton has an energy of 2 GeV. What is its speed?

5. ** Suppose we have a system of two protons colliding head-one so that their total 3-momentum is 0 (also known as the center-of-mass frame). If each has a speed of 0.5 in natural units, what is the invariant mass of the system?

6. ** Below I show the Mandelstam variable s is equal to $(p_1 + p_2)^2$. On page 39, the Mandelstam variable s is also said to equal $(p_3+p_4)^2$. Prove this by using the same technique I used below, but acting on the right vertex. Also, notice that if momentum is conserved at each vertex, then it is ultimately conserved between the incoming and outgoing particles.

7. Problem 2.12 in the book.

8. ** Following the proof in the book for 3D, prove the continuity equation for a one-dimensional system with coordinate x. Namely,

$$\partial_x j + \partial_t \rho = 0 \tag{1}$$

where

$$j = \frac{1}{2im}\left(\psi^* \partial_x \psi - \psi \partial_x \psi^*\right). \tag{2}$$

The 1-D Schrodinger equation to use is given at the bottom of page 41 and is

$$i\partial_t \psi(x,t) = -\frac{1}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t). \tag{3}$$

2 Chapter 2.1 – Units

Many physical predictions are **ratios**. So, if you predict that an object will have a length of 3 meters, what you mean is that the length ratio between that object and a standard meter-stick is "3". "Units" (like 'meters') are a way of keeping track of baselines used for obtaining ratios. In field theory, this concept is particularly important to remember, because ratios can be finite even if the numerator and denominator both diverge.

- Natural units are $\hbar = c = \epsilon_0 = \mu_0 = 1$.
- When using natural units, energy is given in Giga-electron-volts (GeV).
- 1 GeV = 1.6 x 10^{-10} J.
- In natural units $E = m$, so particle masses are also given in GeV.

3 Chapter 2.2 – Special relativity

- The symbol β is defined as v/c but in natural units $\beta = v$.
- The symbol γ is defined as $\frac{1}{\sqrt{1-\beta^2}}$, or more familiar $\frac{1}{\sqrt{1-v^2}}$.
- If observer **A** has coordinates t,x,y,z and if observer **B** – with coordinates t',x',y',z' – is moving at velocity v in the positive z direction with respect to observer **A**, then the coordinates are related by

$$x' = \mathbf{\Lambda} x \tag{4}$$

where the 4-vector x is defined as

$$x = \begin{pmatrix} \mathrm{t} \\ \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix}, \tag{5}$$

the 4-vector x' is defined as

$$x' = \begin{pmatrix} \mathrm{t}' \\ \mathrm{x}' \\ \mathrm{y}' \\ \mathrm{z}' \end{pmatrix} \tag{6}$$

and the Lorentz transformation $\mathbf{\Lambda}$ is defined as

$$\mathbf{\Lambda} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{pmatrix}. \tag{7}$$

Notice that Thomson writes β instead of v in his Lorentz transformation matrices. In natural units (which we are using) they are the same. I've used v since that notation might be more familiar to some of you.

One more thing. The term *4-vector* connotes two things:

- It's an object with 4 components and which combines with other such objects following the usual addition and scalar-multiplication rules of vector spaces.
- It's an object which transforms from one frame to another by a Lorentz transformation as shown. **This part of the definition of 4-vector is not part of the standard mathematics definition of vectors in a vector space.**

- Whenever possible, we compute quantities which are the same in all *inertial* frames (unaccelerated observers moving at constant velocities relative to one another). These quantities are known as **scalars** and are said to be *Lorentz invariant* or simply *invariant*. 4-vectors are **not** scalars.

- If x and y are 4-vectors, then the following object is invariant:

$$x \cdot y \equiv x^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} y. \tag{8}$$

That is, $x \cdot y = x' \cdot y'$. Also, $x \cdot x$ is invariant. If we write this out in components, we get

$$\mathrm{t}^2 - \mathrm{x}^2 - \mathrm{y}^2 - \mathrm{z}^2 = \mathrm{t}'^2 - \mathrm{x}'^2 - \mathrm{y}'^2 - \mathrm{z}'^2. \tag{9}$$

- Another common way of writing arrays and Lorentz dot-products, is to use Greek indices and the Einstein summation convention. This is explained in the text and summarized here.
  - A contravariant 4-vector is described as x^μ which has upper indices. If the components of that vector are $(x^0, x^1, x^2, x^3) =$ (t,x,y,z) then the vector x_μ, with lower indices has components $(x_0, x_1, x_2, x_3) =$ (t,-x,-y,-z). The vector x_μ is said to be a covariant 4-vector.
  - In general, the relationship between an upper index and a lower index 4-vector is this: If the index is 0, then the components are the same. If the index is 1, 2 or 3, the components are negatives of one another.
  - If an expression has an index that appears twice, then you sum over those indices. So, for example, $x^\mu x_\mu = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 = x^0 x^0 - x^1 x^1 - x^2 x^2 - x^3 x^3 = \mathrm{t}^2 - \mathrm{x}^2 - \mathrm{y}^2 - \mathrm{z}^2$.
- Another very important 4-vector is the 4-momentum defined as the contravariant 4-vector $p^\mu \equiv \left(\mathrm{E}, \mathrm{p_x}, \mathrm{p_y}, \mathrm{p_z}\right)$. In this expression, the components are also named $(E, \mathbf{p})$. Notice the convention that boldface letters represent 3-vectors.
- For a single particle of mass m traveling – with respect to the observer – with velocity v (not to be confused with the velocity of a different observer), the components obey the relationships $\mathrm{E} = \gamma m$ and $\mathbf{p} = \gamma m \mathbf{v}$. By direct computation we can see, for a single particle, that $p^\mu p_\mu = m^2$. This relationship is sometimes known as the **mass shell** or **dispersion** equation. In components, this equation becomes $E^2 = m^2 + \mathbf{p}^2$. The notation $\mathbf{p}^2$ denotes the 3-vector dot product $\mathbf{p} \cdot \mathbf{p}$. If $\mathbf{v} = 0$, then $\mathbf{p} = 0$ and we get $E = m$, or putting back c, the famous expression $E = mc^2$. The mass m is sometimes called the *rest mass*, to distinguish it from another quantity that is sometimes called 'mass'.
- Very importantly especially in Feynman diagrams, is the observation that for a system of particles, the sum of their individual momenta is also a 4-vector. **However, that system-momentum does NOT obey a single-particle mass-shell or dispersion relation.** That's

why 'virtual particles' in Feynman diagrams aren't 'real particles'. Their momenta represent systems of particles (generally 2 particles). Despite that, people use the suggestive term *invariant mass* for $p^\mu p_\mu$ even when the 'system is not on the mass shell' (i.e. even when the one-particle dispersion equation isn't obeyed).

- The following notation is used: $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Similarly, $\partial^\mu = \frac{\partial}{\partial x_\mu}$. The chain rule can be used to convert between derivatives ∂_μ and ∂'_μ where $\partial'_\mu = \frac{\partial}{\partial x'^\mu}$. Notice that ∂_μ is a lower-index object that transforms as a covariant vector even though (or more precisely 'because') it is defined as a derivative with respect to components of an upper-index (contravariant) vector.

- Regarding notation p^* explained at the bottom of page 38: You may need this later in the text. However, I've never encountered that notation!

3.1 Chapter 2.2.3 – Mandelstam variables

THIS MAY BE SOMETHING NEW WHICH YOU HAVEN'T ENCOUNTERED BEFORE. Recall that we try to make calculations using invariant quantities, rather than components of 4-vectors. *In the specific scattering case of 2 ingoing particles colliding and becoming 2 outgoing particles* there is a convenient set of 3 invariant quantities that characterize the four separate 4-momenta appearing in the scattering problem. **see Figure 2.2.**

- In the diagrams of Figure 2.2, the incoming particles are shown (with arrows) on the left and the outgoing particles are shown (with arrows) on the right. Their 4-momenta are labelled respectively p_1, p_2, p_3, p_4. Don't get confused into thinking the subscripts refer to components. Each p_i is a 4-vector.

- One rule of Feynman diagrams, that I mentioned in last time's notes, is that 4-momentum is conserved at each vertex. We implement that by assigning a value to the quantity q that appears in the diagrams. For example, in the first diagram, the virtual particle (the squiggly line connected at both ends) is connected on the left to a vertex which has incoming momenta p_1 and p_2. By momentum conservation we then know that $q = p_1 + p_2$. In component form this looks like $(\mathrm{q}^0, \mathrm{q}^1, \mathrm{q}^2, \mathrm{q}^3) = (\mathrm{p}_1^0 + \mathrm{p}_2^0, \mathrm{p}_1^1 + \mathrm{p}_2^1, \mathrm{p}_1^2 + \mathrm{p}_2^2, \mathrm{p}_1^3 + \mathrm{p}_2^3)$. Thomson then writes the p components in terms of the single-particle energies and momenta. In the center-of-mass frame, the components $p_1^i + p_2^i = 0$ for $i = 1, ..., 3$ so $q = (E_1^* + E_2^*, 0, 0, 0)$.

- I also mentioned somewhere, that in momentum-space, Feynman diagrams contribute a term in the scattering amplitude which is proportional to $\frac{1}{q^2-m^2+i\epsilon}$ [1] where the mass m characterizes the particle/field representing the line labeled by q. Remember, this is a virtual particle, not on the mass shell, so it doesn't obey the relationship $q^2 = m^2$ (otherwise the term would blow up). In the first diagram, $q = p_1 + p_2$, and therefore, based on the definition in the text of the Mandelstam variable s, we see that $s = q^2$. So the Mandelstam variable s is what appears in the Feynman term for that diagram, i.e., $\frac{1}{s-m^2}$. [2]

- The value of q in the second diagram is ambiguous (in point of fact, a similar ambiguity exists in the first diagram but it was easier to ignore it). For the top vertex, you could imagine that the conservation of momentum would be achieved by setting $p_1 + q = p_3$, which is how you would think of things if the virtual particle was going from the past to the future – thus acting like an incoming virtual particle. But you could also write $p_1 = q + p_3$ which is how you would think of things if the virtual particle was regarded as an outgoing particle. There's no way to disambiguate that, but it doesn't matter. In either case, $q^2 = (p_1 - p_3)^2$ which, you can see from the book, is the same as the Mandelstam variable t. The Feynman diagram contributes $\frac{1}{t-m^2}$. Once you choose the sign of q in a diagram, you need to stick with it, so at the other vertex you'll have no choice (if it's incoming at the top vertex, it will be outgoing at the bottom vertex).

- The first two diagrams are known as s-channel and t-channel diagrams, after the term appearing in the Feynman amplitude. There is also a u-channel diagram (see the definition of the Mandelstam variable u). [3]

[1] I'll usually omit the ϵ but in cases where $q^2 = m^2$, it's useful to remember that the $i\epsilon$ keeps the expression from diverging. In practice, the expression (known as the Feynman propagator) occurs in complex-valued integrals and controls the kinds of complex contours that can be used for evaluating those integrals

[2] You might wonder what would happen if the virtual particle's mass just happened to be equal $\sqrt{s}$. That's a bit subtle and requires a few modifications (maybe you recall an $i\epsilon$ term from a long time ago?). But often, the resultant scattering amplitudes – while not infinite in the case that $\sqrt{s} = m$ – have a peak at that value of s. This is one of the ways new particles are discovered.

[3] In Figure 2.2, the caption says this only applies when there are identical particles. I don't think that's true. The question to address is whether all of the vertices are 'legal' (by which we mean non-zero). Depending on what particles are being shown, the virtual

4 Chapters 2.3.1 and 2.3.2 – The Schrodinger equation and probabilities

Chapter 2.3.1 should be a review for you, but worth reading to make sure you remember the details. Like most authors, Thomson attempts to motivate quantum mechanics. I've never found such motivations to be very compelling. There's a few items maybe worth highlighting.

- In general, a wavefunction $\psi(\mathbf{x}, t)$ can have any shape at time $t = 0$. The only reason we concentrate on plane waves, is that any function can be written as a 'sum' of plane waves – i.e., a Fourier transform.

- If we know that, at $t = 0$, the wave is of the form $e^{i\mathbf{p}\cdot\mathbf{x}}$, then its behavior at other times is determined by the solution of the Schrodinger equation to be
$$\psi(\mathbf{x}, t) = Ne^{i[\mathbf{p}\cdot\mathbf{x} - Et]} \tag{10}$$

- Since ψ is to be interpreted as a probability amplitude, and there is a total probability 1, of finding the particle somewhere, then for each time t
$$\int d^3x \rho(\mathbf{x}, t) = 1 \tag{11}$$
where the probability density ρ is defined as $\rho(\mathbf{x}, t) = \psi^*(\mathbf{x}, t)\psi^(\mathbf{x}, t)$.

  If we substitute a plane wave in this equation, we'll get
$$\int d^3xN^2 = \infty \neq 1. \tag{12}$$
so in practice we resolve this by limiting the particles to live in a very large box of volume V, in which case we end up showing that $N^2 = 1/V$. Alternatively, the plane wave is regarded as distribution, used only in the context of a Fourier transform of, for example, a wave packet. There are various reasonable ways to put all this on a firm mathematical footing. For example, see the note by Mahendra Mallick on plane wave normalization `https://billcelmaster.com/wp-content/uploads/2021/11/PlaneWave_in_QM_v2.pdf`.

  My impression is that Thomson treats the normalization constant differently than most authors, and claims it should be

particle might be a W-boson, which changes flavors. In that case, I don't believe there'd be anything wrong with the u-channel diagram even if the vertex p_1, p_4, q connects the particle whose momentum is p_1 with a different particle (flavor-changed) of momentum p_4.

interpreted as a number density. I'm not sure where Thomson will go with this.

- Thomson uses 'the Schrodinger picture' – at least in this part of the text. In that formalism, the wavefunctions are time-dependent but the operators are time-independent. Field theorists use the 'Heisenberg' picture where wavefunctions are time-independent and operators are time-independent. The two formalisms are equivalent.
- In Thomson's Schrodinger equation he introduces the operator $\hat{V}$. Although in equation (2.19) this preserves the notion that $\hat{V}$ is an abstract operator, acting on an abstract wavefunction, I think this is unnecessarily abstract for this point in the text. If we treat the wavefunction as a function of coordinate-space, then a normal non-relativistic potential – which is already in the form $V(\mathbf{x})$ – acts on the wavefunction as $V(\mathbf{x})\psi(\mathbf{x}, t)$ and nothing is to be gained by using the hat notation on the very bottom of page 41.
- In case you don't remember notation, $\nabla \equiv (\partial_x, \partial_y, \partial_z)$. [4]

In section 2.3.2, Thomson goes into extra detail on the probability interpretation of the wave-function and, in particular, the notion of a current density (harder to interpret in the Heisenberg picture). This section is somewhat motivational for interpreting scattering amplitudes.

Although the particle has to be somewhere in space, and therefore its total probability is 1, its probability in a given volume varies with time so there's a 'flow'. This is a familiar situation in descriptions of fluids and is generally described by a continuity equation. Here, the continuity equation is

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0, \tag{13}$$

where

$$\mathbf{j} = \frac{1}{2im}(\psi^* \nabla \psi - \psi \nabla \psi^*). \tag{14}$$

The continuity equation can be proven by using the Schrodinger equation, as is shown in the text.

- Using the divergence theorem, the vector $\mathbf{j}$, is interpreted as a flux describing the flow of probability-density through a surface.

[4]Incidentally, when we are doing nonrelativistic mechanics, it doesn't matter if indices are lower or upper.

- For a plane wave, $\mathbf{j} = N^2\mathbf{v}$. Note that Thomson puts absolute values around 'N', but the usual convention is to take N to be a positive real number.