Thomson Chapters 2.3.3 – 2.3.5

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1 Exercises

1. One of the most common useful commutation relations is of the form $[\hat{A}, \hat{B}] = \hat{B}$. This relation and others like it, are responsible for much of the operator algebra in quantum mechanics and quantum field theory. Show that $\hat{B}$ 'raises' the eigenvalues of $\hat{A}$ (and hence is called a 'raising operator'). What this means is: if $|\psi\rangle$ is an eigenstate of $\hat{A}$, with eigenvalue λ, then the state $|\psi\rangle_+ \equiv \hat{B}|\psi\rangle$ is an eigenstate of $\hat{A}$, with eigenvalue $\lambda + 1$.

   Hint: If you look at page 48, you'll see an example of how this works.

2. Prove that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$.

   Use eqs. (8,9,12)). Also note that $[\hat{x}, \hat{y}] = [\hat{p}_x, \hat{p}_y] = [\hat{p}_x, \hat{y}] = 0$ etc. In other words, the only non-zero commutators between the position and momentum operators, are the three in eqs. (8,9).

   I'll illustrate the kind of manipulations to be done by deriving $[\hat{y}\hat{p}_z, \hat{z}\hat{p}_x]$.

$$\begin{aligned}
[\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] &= \hat{y}\hat{p}_z\hat{z}\hat{p}_x - \hat{z}\hat{p}_x\hat{y}\hat{p}_z \\
&= \hat{y}\hat{z}\hat{p}_z\hat{p}_x - i\hbar\hat{y}\hat{p}_x - \hat{z}\hat{p}_x\hat{y}\hat{p}_z \\
&= \hat{y}\hat{z}\hat{p}_z\hat{p}_x - i\hbar\hat{y}\hat{p}_x - \hat{z}\hat{y}\hat{p}_x\hat{p}_z \\
&= \hat{y}\hat{z}\hat{p}_z\hat{p}_x - i\hbar\hat{y}\hat{p}_x - \hat{y}\hat{z}\hat{p}_z\hat{p}_x \\
&= -i\hbar\hat{y}\hat{p}_x.
\end{aligned} \tag{1}$$

   The first term on the RHS of the first line, is transformed to the first two terms of the RHS of the second line, using the z-commutator of eq. (9). The third and fourth lines are both obtained by transforming the last term on the RHS (starting with the second line) by commuting operators whose commutator is 0 (see above). Then in the fourth line the first and third line cancel, leaving the fifth line.

2 Chapter 2.3.3 – Time dependence and conserved quantities

This section is self-contained. I'll only highlight a few things but if you're interested in derivations, I think you'll find the text is quite detailed and complete.

- Thomson shifts notation from $\psi(x,t)$ to $|\psi(x,t)\rangle$. This is more or less self-explanatory, but has different mathematical connotations. The first expression $\psi(x,t)$ is mathematically a complex-valued function of the variables (x,t). The second expression is mathematically a vector in a Hilbert space. Actually, the second expression is technically incorrect. It should be $|\psi\rangle$ (or better yet, $|\psi(t)\rangle$). The reason these two expressions can be associated, is that there is a natural way of identifying 'the space of complex functions' with a vector space. The identification is $\psi \leftrightarrow |\psi\rangle$. Addition of functions, corresponds to additions of vectors. But there's more. In quantum theory, it isn't sufficient to speak of the complex functions (aka wave functions). What matters are probabilities, and for that, we ultimately need to deal with expressions like $\int d^3x \psi^*(\mathbf{x},t)\phi(\mathbf{x},t)$. When we identify the wave functions with vectors in a Hilbert space, the integral is regarded as an inner product between ψ and ϕ so $\int d^3x \psi^*(\mathbf{x},t)\phi(\mathbf{x},t) \leftrightarrow \langle\psi|\phi\rangle$.

  The reason for shifting to the vector notation (Dirac bras and kets) is that it allows generalizations and also makes many abstract manipulations much easier. Once we shift to vector notation, then wave-function operations like differentiation are identified as operators on the vectors. The derivative of one function is a different function. In operator language, you operate on one vector to produce a different vector, so for example you write $|\psi\rangle \to \hat{A}|\psi\rangle$.

- Thomson also uses the ambiguous notation $\langle\hat{A}\rangle$ to mean $\langle\psi|\hat{A}|\psi\rangle$ (the expectation value of the operator $\hat{A}$ in the presence of state $|\psi\rangle$). It's ambiguous, because the left-hand side makes no reference to the state $|\psi\rangle$, and its value most definitely depends on that state. Often, but not always, the state can be determined by context, and for abstract theorems, the state often doesn't matter. Still, it's bad practice to use Thomson's notation.

- Having introduced Dirac vector notation, Thomson then regresses, when deriving expressions like eq. (2.28), to a hybrid wave-function / vector approach (the explicit integrals of wave-functions, but still representing operations with the abstract operator formalism). There is, in fact, a completely self-consistent way of using a pure vector approach. However, I'd argue this would obfuscate the derivation and if you feel reasonably comfortable with Thomson, stick with it. Just treat objects like $\hat{A}$ as operations (like a derivative, or multiplication by a scalar) on wave functions.

- Another aspect of Thomson's derivations, is that the operators are not explicitly time-dependent. What that means is this. You can write the operators as functions of other quantities (such as other operators) but you should not have an independent time variable. For example, the following is NOT OK ... $\hat{A}(\hat{x}, t)$. From the point of view of the physical laws for a complete system, this restriction follows from a statement that, in the Schrodinger picture 'observables are represented by Hermitian operators that have no explicit time-dependence'. [1]

- Finally, we get to equation (2.29), which says that the expected value of an operator changes with time proportionally to the expectation value of its commutator with the Hamiltonian. Importantly, the equation holds for every state $|\psi\rangle$ used to construct the expectation value (remember my above comments about the ambiguity of the notation $\langle\rangle$ when expressing operator expectation values) . In my opinion, it would be best to write eq. (2.29) as

$$\frac{d\langle\hat{A}\rangle_{|\psi(t)\rangle}}{dt} = i\langle[\hat{H}, \hat{A}]\rangle_{|\psi(t)\rangle} \tag{2}$$

When the LHS is 0, we say that $\hat{A}$ is conserved. Conserved observables are those that commute with the Hamiltonian.

- I found particularly interesting the derivation at the bottom of page 44. Maybe I knew this once, but if so, I'd forgotten it. If the state $|\psi\rangle$ happens to be an eigenstate of the Hamiltonian, then, in that state, the expectation value of EVERY observable is time-independent. That's pretty cool.

- Thomson eq. (2.30) accompanies an almost-throwaway comment about neutrino oscillations. This is worthy of slightly more elaboration. What

[1] It's worth noting that if one is describing a partial system, then there can be external forces which manifest themselves, in the partial system, as having explicit time-dependencies.

Thomson shows in eq. (2.30) is that **if** you can prepare **a state** $|\psi\rangle$ **which is a superposition of energy eigenstates – let's say two eigenstates – then that state will evolve as follows:**

$$|\psi(\mathbf{t})\rangle = \mathbf{c_1}|\phi_\mathbf{1}\rangle\mathbf{e}^{-\mathbf{iE_1t}} + \mathbf{c_2}|\phi_\mathbf{2}\rangle\mathbf{e}^{-\mathbf{iE_2t}} \tag{3}$$

If you then look at the expectation values of certain operators, these will oscillate owing to cross terms that look like $\cos[(E_1 - E_2)t]$. In various nuclear processes occurring on stars, the states produced are a superposition of electron and muon neutrino energy (aka 'mass') eigenstates, and thus certain observations exhibit the oscillations. The reason why these superpositions are created, is that the Lagrangian involves fields often called the 'electron neutrino field' or 'muon neutrino field', but which are coupled to one another via the weak interaction terms in the Lagrangian, and therefore create cross-terms in the Hamiltonian so that mass (energy) eigenstates are linear combinations of the particle-states associated with each field.

3 Chapter 2.3.4 – Commutation relations and compatible observables

This section is also self-contained. Here are the key conclusions. One important theorem is implicit, but I haven't noticed it in the text. Namely, **if $\hat{A}$ is a Hermitian operator, then there is a basis of states $|\phi_i\rangle$ meaning that any state $|\psi\rangle$ in the Hilbert space can be written as a superposition of those basis states. Namely,**

$$|\psi\rangle = \sum_i c_i|\phi_i\rangle. \tag{4}$$

- Suppose $\hat{A}$ and $\hat{B}$ are two operators which commute. That is,

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = 0. \tag{5}$$

  Then there is a basis of states $|\phi_i\rangle$ so that they are eigenstates of both $\hat{A}$ and $\hat{B}$. That is

$$\begin{aligned} \hat{A}|\phi_i\rangle &= a_i|\phi_i\rangle \\ \hat{B}|\phi_i\rangle &= b_i|\phi_i\rangle \end{aligned} \tag{6}$$

  In such a case, it is common to write $|a_i, b_i\rangle$ instead of $|\phi_i\rangle$ and we refer to a_i and b_i as quantum numbers. They signify *the measured values, for that state, of the observables represented by operators $\hat{A}$ and $\hat{B}$.*

- Suppose $\hat{A}$ and $\hat{B}$ are two operators which do **not** commute. Then

$$\Delta_{|\psi\rangle}\hat{A}\Delta_{|\psi\rangle}\hat{B} \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle_{|\psi\rangle}|, \tag{7}$$

where $(\Delta_{|\psi\rangle}\hat{A})^2 = \langle\hat{A}^2\rangle_{|\psi\rangle} - \langle\hat{A}\rangle^2_{|\psi\rangle}$. This statement isn't proven. It's significance is illustrated by examining the non-commuting pair of operators $\hat{x}$ and $\hat{p}_x = -i\frac{\partial}{\partial x}$.

- Thomson shows that

$$[\hat{x},\hat{p}_x] = +i\hat{\mathbf{I}} \tag{8}$$

and it can similarly be shown that

$$\begin{aligned} [\hat{y},\hat{p}_y] &= +i\hat{\mathbf{I}} \\ [\hat{z},\hat{p}_z] &= +i\hat{\mathbf{I}} \end{aligned} \tag{9}$$

where $\hat{\mathbf{I}}$ is the operator which multiplies a state by 1. Often, this operator is dropped since it is implied by context. Then by applying eq. (7) above, we immediately deduce that

$$\Delta_{|\psi\rangle}\hat{x}\Delta_{|\psi\rangle}\hat{p}_x \geq \frac{1}{2}. \tag{10}$$

Although this equation has been derived with natural units, it's common to reinsert $\hbar$. Also, since the inequality is true for all states, it's unnecessary to specify the state $|\psi\rangle$ as a subscript. Finally, since the 'hat' notation isn't universally used, we can drop the 'hats'. We end up with the familiar Heisenberg uncertainty principle

$$\Delta x\Delta p_x \geq \frac{\hbar}{2}. \tag{11}$$

The equation is interpreted as meaning that it isn't possible to simultaneously measure the values of position and momentum. The magnitude of uncertainty is on the order of the Planck constant $\hbar$ and thus only affects measurements that are small – i.e., on the quantum scale. From eq. (7), this uncertainty is characteristic of all pairs of non-commuting observables (Hermitian operators).

By contrast, we showed above that for commuting operators, there are eigenstates such that one can measure exact values for both the operators.

4 Chapter 2.3.5 – Angular momentum in quantum mechanics

This section illustrates concretely the general principles of conservation and compatible observables from the previous sections. Recall some of the key results.

- If an observable represented by the Hermitian operator $\hat{A}$ commutes with the Hamiltonian, i.e., $[\hat{A}, H] = 0$, then the observable is conserved. That is, the observable's expectation value doesn't change with time.

- If two Hermitian operators $\hat{A}$ and $\hat{B}$ commute, then there is a complete set of states (meaning that any state can be written as a linear combination of members of the complete set) that are eigenvectors of both operators. If, for one of those states $|\psi\rangle$, we have $\hat{A}|\psi\rangle = a|\psi\rangle$ and $\hat{B}|\psi\rangle = b|\psi\rangle$, then we write the state as $|a, b\rangle$ and we say that a and b are simultaneously measured values of $\hat{A}$ and $\hat{B}$.

We now consider the angular momentum operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$ and $\hat{\mathbf{L}}^2 \equiv \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. We've discussed these operators many times in different contexts. You may remember that they are related to the operations of rotation, and that these operations can be used to constrain equations of motion that have rotational symmetry. In quantum theories whose states can be represented by complex-valued scalar (i.e., one component) wave functions, we define $\hat{L}_x, \hat{L}_y, \hat{L}_z$ by

$$\begin{aligned} \hat{L}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{aligned} \tag{12}$$

where $\hat{p}_z = -i\hbar\frac{\partial}{\partial z}$ etc.

The text shows the commutation relations amongst the various angular momentum operators. All of them can be derived by brute force (see the exercises at the beginning of these notes). Previously when we encountered the angular momentum operators as generators of rotations, we discovered these same commutation relations and discovered they were properties of the Lie Algebra of the rotation group.

Let's connect these operators to the general discussions of the past sections.

- Each of $\hat{L}_x, \hat{L}_y, \hat{L}_z$ and $\hat{\mathbf{L}}^2$, commutes with the Hamiltonian, provided the potential is spherically symmetrical (i.e., V only depends on r). So angular momentum is conserved. Interestingly, Thomson neither shows this nor even mentions it. However, it's probably the most important fact about those operators. In classical mechanics, angular momentum is a conserved quantity. That's also true in quantum mechanics.

- Thomson shows that the eigenvalues of the operator $\hat{\mathbf{L}}^2$ can be written as $l(l+1)$ where l is an integer (we are suppressing the $\hbar$). He also shows that the eigenvalues of the operator $\hat{L}_z$ can be written as m where m is an integer.

- Each of $\hat{L}_x, \hat{L}_y, \hat{L}_z$ commutes with $\hat{\mathbf{L}}^2$. So, for example, it's possible to measure simultaneously the z-angular momentum and $\hat{\mathbf{L}}^2$. We therefore characterize states as $|E, m, l\rangle$, where E is an eigenvalue of the Hamiltonian (i.e. the state's energy), m is an eigenvalue of $\hat{L}_z$, and $l(l+1)$ is an eigenvalue of $\hat{\mathbf{L}}^2$.

- $\hat{L}_x$ doesn't commute with $\hat{L}_z$, etc. Therefore we can't simultaneously measure the x and z angular momenta. We could have chosen to characterize states using the eigenvalues of $\hat{L}_x$ or $\hat{L}_y$, both of which have integer eigenvalues just like $\hat{L}_z$. However, the standard convention is to characterize states using the eigenvalues m of $\hat{L}_z$ as we did above.