Clarification on Fermi's Golden Rule

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1 Fermi's Golden Rule

Reference

Eisberg, Robert, and Robert Resnick, *Quantum physics of atoms, molecules, solids, nuclei, and particles*, John Wiley, Second Edition, 1985.

Appendix K: TIME-DEPENDENT PERTURBATION THEORY, pp. K-1-- K-5 gives a clear description.

On p. K-4 it clearly states:

" … the number of final quantum states dN_f per energy interval dE_f is the density of final states

$\rho_f = dN_f / dE_f$."

I have used our notation convention from Thomson's book for convenience.

Key Steps

Given that the particle was in the state $\phi_i(\mathbf{x})$ at time $t = 0,$ we have

$$\begin{aligned} c_f(T) &= \frac{-1}{(E_f - E_i)} H'_{fi} \left[e^{i(E_f - E_i)T/\hbar} - 1 \right] \\ &= \frac{-1}{(E_f - E_i)} T_{fi} \left[e^{i(E_f - E_i)T/\hbar} - 1 \right]. \end{aligned} \tag{1-1}$$

Remark: I didn't use Thomson's (2-45), since this can be easily integrated. There is no need to consider the double integral after (2-45).

Then the probability for a transition to the state $\phi_f(\mathbf{x})$ at time $t = T$ is given by

$$T_{fi} := c_f^*(T) c_f(T). \tag{1-2}$$

We can show that

$$T_{fi} := c_f^*(T) c_f(T) = \frac{1}{\hbar^2} \left| T_{fi} \right|^2 \frac{\sin^2(\beta_{fi} T)}{\beta_{fi}^2}, \tag{1-3}$$

where

$$\beta_{fi} = (E_f - E_i) / 2\hbar. \tag{1-4}$$

From Eisberg & Resnick: The perturbation $H'(\mathbf{x},t)$ has the effect of mixing in contributions from other states over a whole range of the quantum number *f*. However, we see that the most important contributions come from those f which correspond to eigenvalues E_f lying within a range centered about E_i and of width ΔE, where

$$\Delta E \sim 2\pi\hbar / T. \tag{1-5}$$

The **transition probability** P_i from the initial state $\phi_i(\mathbf{x})$ to all other states at *T* is defined as

$$P_i := \sum_{\substack{f \\ f\neq i}} c_f^*(T)c_f(T). \tag{1-6}$$

From Eisberg & Resnick**: Important**: To evaluate it, we assume that there are a large number of closely spaced final quantum states in the range ΔE; the number of final quantum states dN_f per energy interval dE_f is the density of final states $\rho_f = dN_f / dE_f$. That is

$$P_i \simeq \int_{-\infty}^{\infty} c_f^*(T)c_f(T)\,dN_f = \int_{-\infty}^{\infty} c_f^*(T)c_f(T)\frac{dN_f}{dE_f}dE_f, \tag{1-7}$$

$$P_i = \int_{-\infty}^{\infty} c_f^*(T)c_f(T)\,\rho_f(E_f)dE_f. \tag{1-8}$$

Substitution of (1-3) and (1-4) in (1-8) gives

$$P_i \simeq \frac{1}{\hbar^2}\int_{-\infty}^{\infty} \left|T_{fi}\right|^2 \frac{\sin^2([(E_f - E_i)/2\hbar]T)}{[(E_f - E_i)/2\hbar]^2}\rho_f(E_f)dE_f. \tag{1-9}$$

If we assume that the matrix element T_{fi} and the density of final states ρ_f are both slowly varying functions of E_f in the range ΔE, then we get

$$P_i \simeq \frac{2\pi}{\hbar}\left|T_{fi}(\bar{E}_f)\right|^2 \rho_f(\bar{E}_f)T. \tag{1-10}$$

Thus, the transition probability is proportional to *T*, as expected. The **transition rate** R_i is define by

$$R_i := \frac{dP_i}{dT}. \tag{1-11}$$

Hence the **transition rate** is given by

$$R_i \simeq \frac{2\pi}{\hbar}\left|T_{fi}(\bar{E}_f)\right|^2 \rho_f(\bar{E}_f), \tag{1-12}$$

where $\bar{E}_f$ is the average energy in the interval $\Delta E.$ We note from (1-12) that the transition rate is independent of *T*