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Dec 1, 2021

1 Problems 1 and 2

Problem 1. Given $B|\psi\rangle = \lambda|\psi\rangle$ and $[B, A] = A$, show that $BA|\psi\rangle = (1+\lambda)A|\psi\rangle$.

$$B|\psi\rangle = \lambda|\psi\rangle. \tag{1-1}$$

$$[A, B] = B. \text{ Incorrect} \tag{1-2}$$

From Thomson (2-33), p. 48, we have

$$[L_z, L_+] = L_+. \tag{1-3}$$

$$[B, A] = A. \text{ Correct} \tag{1-4}$$

$$BA - AB = A. \tag{1-5}$$

$$BA = A + AB. \tag{1-6}$$

$$\begin{aligned} BA|\psi\rangle &= (A + AB)|\psi\rangle \\ &= (A + \lambda A)|\psi\rangle \\ &= (1+\lambda)A|\psi\rangle. \end{aligned}$$

Hence

$$BA|\psi\rangle = (1+\lambda)A|\psi\rangle. \tag{1-7}$$

Thus $A|\psi\rangle$ is an eigenstate of B with eigenvalue $1+\lambda$.

Problem 2. Prove that $\left[L_x, L_y\right] = i\hbar L_z$.

We have

$$\begin{aligned} L_x &= yp_z - zp_y, \\ L_y &= zp_x - xp_z, \\ L_z &= xp_y - yp_x. \end{aligned} \tag{1-8}$$

$$[x, p_x] = i\hbar, \;\; [y, p_y] = i\hbar, \;\; [z, p_z] = i\hbar. \tag{1-9}$$

Expanding $\left[L_x, L_y\right]$, we get

$$\begin{aligned} \left[L_x, L_y\right] &= \\ &= \left[yp_z - zp_y, zp_x - xp_z\right] \\ &= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z] \end{aligned} \tag{1-10}$$

Expanding the first term in (1-10), we get

$$[yp_z, zp_x] = yp_z zp_x - zp_x yp_z. \tag{1-11}$$

From the commutator of z and p_z, we have

$$\begin{aligned} zp_z - p_z z &= i\hbar, \\ p_z z &= zp_z - i\hbar. \end{aligned} \tag{1-12}$$

Using (1-12) in (1-11), we get

$$\begin{aligned} [yp_z, zp_x] &= y(zp_z - i\hbar)p_x - zp_x yp_z \\ &= yzp_x p_z - i\hbar yp_x - yzp_x p_z \\ &= -i\hbar yp_x. \end{aligned}$$

Hence

$$\boxed{[yp_z, zp_x] = -i\hbar yp_x.} \tag{1-13}$$

Expanding the second and third terms in (1-10), we get

$$\begin{aligned} [yp_z, xp_z] &= yp_z xp_z - xp_z yp_z \\ &= xyp_z p_z - xyp_z p_z = 0. \end{aligned} \tag{1-14}$$

$$[zp_y, zp_x] = zp_y zp_x - zp_x zp_y = 0. \tag{1-15}$$

Expanding the fourth term in (1-10) and using (1-12), we get

$$\begin{aligned} [zp_y, xp_z] &= zp_y xp_z - xp_z zp_y \\ &= xzp_y p_z - x(zp_z - i\hbar)p_y \\ &= xzp_y p_z - xzp_z p_y + i\hbar xp_y \\ &= i\hbar xp_y. \end{aligned}$$

Hence

$$\boxed{[zp_y, xp_z] = i\hbar xp_y.} \tag{1-16}$$

Substituting (1-13)-(1-16) in (1-10), we get

$$\left[L_x, L_y\right] = -i\hbar yp_x. + 0 + 0 + i\hbar xp_y = i\hbar(xp_y - yp_x) = i\hbar L_z.$$

Hence,

$$\boxed{\left[L_x, L_y\right] = i\hbar L_z.} \tag{1-17}$$