Thomson Chapters 2.3.6 – 3.1. Fermi's golden rule

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December 16, 2021

1 What do we measure in experiments?

- Much of (quantum) physics is characterized by systems which are mostly simple and free, but which occasionally and briefly have small interactions.
- The two primary situations studied in QFT are:
  - Decay: One particle mostly does nothing ('simple and free') but occasionally deteriorates into other particles, which then behave freely (travel in straight lines at constant velocity – as wave packets).
  - Scattering: Two particles mostly travel in straight lines of constant velocity (wave packets) but may briefly interact (collide) with one another during which time they might transform into other particles. After the collision, all the particles again travel in straight lines (wave packets).
- In both cases, the initial and final configurations are described as initial and final 'states'. Those initial and final states, if nonrelativistic, are described by the multiparticle Schrodinger equation for free particles. Each particle's wave packet obeys (in natural units)

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} \tag{1}$$

  which is easily solved and describes 'quantum straight line motion'.

  For relativistic quantum theory, similar kinds of equations describe free particle motion. **We say they are governed by a free Hamiltonian H_0**

- The question we ask, for both decay and scattering, is "if we have beams of particles each of which is in the same initial state, then what is the rate of transitioning into a particular final state?" For example, if we have a beam of neutral kaons, what is the rate of decay into two pions flying apart – in the rest frame – along the z axis?

- The above example illustrates one subtlety of all this. Since the pions could fly off in any direction, then the probability (and thus the rate) of going in *exactly* the z direction is 0. A better question would be "what is the rate that the pions will fly apart within an angle of 10 degrees of the z-axis?" That rate is definitely non-zero. This example shows us that the quantity of interest is really a **rate density**. However, we call that **the rate**.

- Again with the same example, rotational symmetry tells us that the appropriate way to integrate probability density (in the lab frame), is with an integration measure that is the same in all directions.

2 How are our measurements related to the Hamiltonian?

- We mathematically describe the above physical situations by a Hamiltonian which is written as a sum

$$H = H_0 + H_I \tag{2}$$

where H_0 is called the 'free Hamiltonian' and H_I is called the 'interaction Hamiltonian' (actually, I prefer to call it the 'interactive part of the Hamiltonian').

- For the purposes of almost everything we do in QFT, H_0 is a Hamiltonian describing particles traveling freely, and H_I is a term describing interactions small enough so that perturbation methods can be applied.

- Although particle beams are best described by wave packets, it's convenient to Fourier-decompose those packets into momentum eigenstates sometimes called **plane waves**. In practice, particle beams are prepared and measured at fairly precise momenta. We'll shortly see how this matters.

- What follows is an **approximation** to an exact prediction.

- We'll start with non-relativistic QM. The principles are the same for relativistic QM (Quantum Field Theory).
- We'll pretend that the period of interaction is finite. That's approximately true since, in collisions, the interactions occur only when the beams more or less cross one another. We use this approximation by saying that the interaction Hamiltonian is 'turned on' at a certain time, say $T = 0$. In decays, the situation is somewhat different but we can make the same pretense.
- For convenience we'll take $H_I(t) = 0$ when $T < 0$ but otherwise is a constant. .
- We'll express our predictions in terms of initial and final states that are eigenstates of the free Hamiltonian. Our prediction will be obtained as "the probability density that, starting with the system in the specified initial state, it will end up after a time T in the specified final state". We divide by T to obtain the rate.
- In practice, the rate is measured by computing the number of particles are in the colliding beams over time T, then measuring how many particles are in a particular final state, and then dividing by T.
- **Fermi's Golden Rule**

  $$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i) \tag{3}$$

  where Γ_{fi} is the rate at which the initial state i transitions to the final state f, $T_{fi} \equiv \langle f|H_I|i\rangle$ and is called the transition matrix, and ρ is the density of final states for that initial state. This is explained more fully in the next section.
- **THE MAIN TAKEAWAY IS THAT SCATTERING AND DECAY PREDICTIONS REQUIRE TWO KEY INGREDIENTS**.
  1. **The transition matrix – which we compute in QFT with Feynman diagrams**
  2. **The density of states – which is primarily a kinematic factor that can be computed exactly independent of the Hamiltonian.**

  The density of states is the primary topic of Thomson Chapter 3.
- **Less important for us** Fermi's Golden Rule (which is closely related to the *Born approximation*) is a first-order perturbation

expansion. The next-order is obtained by expanding the transition matrix:

$$T_{fi} = \langle f|H_I|i\rangle + \sum_{k\neq i} \frac{\langle f|H_I|k\rangle\langle k|H_I|i\rangle}{E_i - E_k} \tag{4}$$

3 The density of states

In the original version of these notes, I followed Thomson's notation in which he parametrized the density of states by the initial energy. However, after our discussion, I've concluded that I don't completely understand what Thomson is saying. It looks somewhat wrong to me, but I may be missing something. For example, the 'accessible' final states could depend on the initial state if the initial state had some conserved quantum numbers that would prohibit transitions to certain final states. Usually, this kind of thing would be taken care of by the transition matrix which would be 0, but the example illustrates that one must be careful of too much abstraction.

In any case, other sources appear to parametrize the density by the final energy – which feels more likely to cover cases of interest. For more details, see Mahendra's notes on this.

- In Thomson Chapter 2, the simplest of situations is considered. There, the initial and final states are characterized completely by their energy.
  - The density of states for energy E_f is defined by saying that the number of accessible energy states in an interval E_f to $E_f + \delta$ is $\rho(E_f)\delta$. By an accessible energy state, we mean 'a state $\langle s|$ which can be transitioned-to from the initial state described by $|E_i\rangle$'. In other words $\langle s|H_I|E_i\rangle \neq 0$.
  - More importantly, an assumption is made, that T_{fi} is highly insensitive to the precise value of f. So above, we'd have $\langle s|H_I|E_i\rangle = \langle E_f|H_I|E_i\rangle$.
- More generally as we proceed to particle physics, the initial state is given by the 4-momenta of each particle in the beam, and the density of final states is defined with respect to an interval around a particular configuration of final 4-momenta.