Some basic relativistic quantum mechanics puzzles

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1 Revision History

- January 4: Laid out basic issues. Couldn't find anything wrong with the notion of a Klein-Gordon wavefunction, provided energy > 0. Couldn't construct a good reason why this condition couldn't be imposed by fiat.

- January 5: Found that negative energy was required in order to avoid propagation outside the light cone. Leads to different speculations about probability current. Material was rewritten to accommodate all this.

- January 5.5: Perhaps integrals over momentum should be modified for Lorentz-invariance. That has consequences for the interpretation of the position-eigenstates and in particular, they aren't a complete orthonormal set. Material has been added to consider this.

2 Overview

During our January 3 meeting, two issues were raised that I more or less swept under the rug. After the meeting, I realized I've been sweeping these under the rug forever, and that it was time for me to look under the rug. Unfortunately, I haven't come to any completely satisfactory conclusions but I thought it would be worth laying out the things that are bothering me. I'm hoping someone will look into this either by staring the problems into submission or by looking online for someone who has a relevant treatment of the subject. For now, I don't know whether my issues are trivial or deep. I don't think they are mathematically very challenging. Rather, I believe my issues are interpretive.

The two subject areas are:

- The 'time-energy' Heisenberg uncertainty principle
- The appropriate interpretation of wave functions, probability densities and currents for electrons and other particles arising from field theory

Although the topics are separate from one another, I'll discuss the second topic by building on some of my notes for the first topic.

Any and all comments are welcome!

3 Time-energy uncertainty principle

Heisenberg's uncertainty principle was introduced in the context of non-relativistic QM. *For simplicity, imagine we are in one spacial dimension.*

- The statement of the uncertainty principle is this: if we have a way of measuring a particle's momentum and if we also have a way of measuring the particle's position, then it's not possible to simultaneously measure both with complete accuracy. In particular,

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{1}$$

  where Δ denotes the variance measured when repeating the experiments (technically when repeating them an infinite number of times).

- The derivation of this inequality arises from the fact that in non-relativistic QM, the position of a particle is related to its momentum via a Fourier transform. If the amplitude for a particle to have momentum p is $\tilde{\psi}(p)$ and if the amplitude for that particle to have position x is $\psi(x)$, then according to non-relativistic QM (by the way, I'm setting $\hbar$ to 1).

$$\psi(x) = \int \frac{dp}{2\pi} e^{ipx} \tilde{\psi}(p). \tag{2}$$

  Then as Matthew mentioned, there is a fact about Fourier transforms where the x-variance times the p-variance is bounded from below by a constant (I didn't do the math but feel free. I'm hoping you'll see that the constant is $\frac{1}{2}$).

- Furthermore, all of this is cast in algebraic form by defining the operators $\hat{x}$ and $\hat{p} = -i\partial_x$, then applying the commutation relations etc.

- In non-relativistic QM, time is regarded as a parameter **and not an operator**. However, the energy **is** regarded as an operator, generally known as H or $\hat{H}$, the Hamiltonian operator.

- Yesterday, the following natural question came up. We know from the theory of relativity, that time and position should be different perspectives on the same thing. Similarly for energy and momentum. So wouldn't it be natural, in anticipation of a relativistic theory, that there should be an uncertainty principle between time and energy?

- At first blush, the answer is "no" since there is no operator corresponding to time. Time is simply a parameter in the non-relativistic theory.

- HOWEVER, time and energy are related, in non-relativistic QM (and even in relativistic QM) by a Fourier transform operation. The argument goes like this: Any (free-particle) state can be decomposed into a superposition of energy basis states

$$\psi(t) = \int \omega(E) e^{-iEt} dE, \tag{3}$$

  For simplicity, I've assumed we're only looking at position $x = 0$ so I've suppressed the x-variable. e^{-iEt} is the free-particle eigenstate with eigen-energy E and $\omega(E)$ is the coefficient in the superposition. (Also, for simplicity I'll assume non-degenerate energy states and if you don't know what that means, don't worry about it.)

- Just as before, from the Fourier transform theory, the variance of t multiplied by the variance of E is bounded from below by a constant (proportional to $\hbar$).

- As an example, imagine $\omega(E) =$ constant. Then the integral is proportional to a delta function around 0. So if ω has an infinite variance then the wave function spikes at time = 0. (There's nothing at $x = 0$ before time 0 and nothing after). Conversely, if ω is narrowly peaked at some energy E_0, then the wavefunction will be almost a plane wave of the form $e^{-iE_0 t}$ – which has infinite variance.

So we see there's a time-energy uncertainty principle despite the fact that there isn't a time operator and therefore no 'time-Hamiltonian' commutation relationship.

THE QUESTIONS WHICH PUZZLE ME: In non-relativistic theory, what is the structural similarity between position and time that leads to the time-energy uncertainty principle? Could we invent a time operator and define it in a similar way to the position operator as $\hat{t}|\psi, t\rangle = t|\psi, t\rangle$? If not, why not? If so, why doesn't anyone do that? Could the existence of a time-energy uncertainty principle lead pre-relativistic physicists to suspect that time and position should be on equal footing? (Mind you, all QM physicists after Planck were 'post-relativistic'.)

In case you are wondering: Relativistic QM didn't evolve by making time an operator on the same footing as position. Instead it made position a parameter on the same footing as time. So traditionally, one doesn't speak of position operators in relativistic QM. **This may be especially relevant to the next section.** Only recently (in the past 30-40 years) has this all been re-visited. In string theory, position and time become operators and the 'parameters' are (unphysical) ways of identifying particular combinations of position and space as potential values of 'events'.

4 Wave functions in Quantum Field Theory

The point of view I'll adopt here is that the only relativistic QM we study, is QFT. (As I mentioned above, another variant of relativistic QM is string theory – which can be formulated as a QFT but differently than we've been doing.) In particular, even though we treat the Dirac equation as a relativistic version of the Schrodinger equation, the proper relativistic theory of the electron is QFT.

One immediate consequence of all this, is that position is treated as a parameter, just as time was treated in the non-relativistic case. **QFT (and its manifestation in the Dirac theory of the electron) does not discuss position operators!** I'll return to this point, but first must point out that without position operators it's questionable what position wavefunctions should mean in relativistic QM.

As I'll discuss, we can still define a quantity like $\psi(x)$ but there might be issues with its interpretation. On the other hand, in relativistic QM, it **is** perfectly reasonable to discuss quantities like $\psi(p)$ and to interpret these as the probability amplitude that the state is measured to have a momentum of p. That's because momenta, in QFT, **are** observables (operators representing measurement).

This discussion is a bit reminiscent of the previous section where I raised the question of how the energy – an operator – should be related to time, which is a parameter and not an operator. With all this as a preface, here are a series of points to think about.

- For the remainder of this discussion, consider the relativistic description of a massive scalar particle like a pion. The particle has no spin so is completely characterized by its 3-momentum so is depicted $|\mathbf{p}\rangle$.

- Furthermore, since we are dealing with a relativistic theory, the energy, E is related to the momentum by the equation

$$E = \sqrt{p^2 + m^2} \tag{4}$$

- In general, the Hilbert space is a multi-particle space (a tensor product of one-particle states) but we'll focus on the single-particle states. The states $|\mathbf{p}\rangle$ would form a complete set if the particle-momentum was completely sufficient to determine all properties of the particle including how it propagates in time. I'll begin the analysis by assuming a complete set **but be warned that I'll run into a critical problem which will force me to introduce another set of momentum states distinguished by a quantity having to do with propagation properties.**

- **CAVEAT EMPTOR: In what follows, I'm going to be very careless about normalization and factors of 2π. The qualitative results won't depend on that carelessness.**

- For the time being, continue to assume that the states $|\mathbf{p}\rangle$ form a complete set. That is, any single-particle state can be written as a superposition,

$$|\psi\rangle = \int \psi(\mathbf{p})|\mathbf{p}\rangle d^3p, \tag{5}$$

  where the complex coefficients $\psi(\mathbf{p})$ are called "the p-representation of $|\psi\rangle$" or more simply "the momentum wavefunction of $|\psi\rangle$".

- Now define a new set of states as follows:

$$|\mathbf{x}\rangle = \int d^3p e^{i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle. \tag{6}$$

  These form a complete set of states (again, that assumes the momentum states were a complete set) and it's tempting to call them "position eigenstates". However, that would be appropriate only if we can identify a position observable. We'll return to this point shortly.

- Construct the inner product of $|\psi\rangle$ and $|\mathbf{x}\rangle$ as

$$\begin{aligned}\langle \mathbf{x}|\psi\rangle &= \int d^3p' d^3p e^{-i\mathbf{p}'\cdot\mathbf{x}}\psi(\mathbf{p})\langle \mathbf{p}'|\mathbf{p}\rangle \\ &= \int d^3p' d^3p e^{-i\mathbf{p}'\cdot\mathbf{x}}\psi(\mathbf{p})\delta(\mathbf{p}'-\mathbf{p}) \\ &= \int d^3p e^{-i\mathbf{p}\cdot\mathbf{x}}\psi(\mathbf{p})\end{aligned} \tag{7}$$

  The RHS is just the Fourier transform of $\psi(\mathbf{p})$ so it would be natural to call this the x-wavefunction $\psi(\mathbf{x})$. [1]

- **THE KEY QUESTION IS: "what is the significance of $\langle \mathbf{x}|\psi\rangle$?"** Sure, we can write this as $\psi(\mathbf{x})$, but what does it mean? In non-relativistic QM, that quantity would be interpreted as the probability amplitude that the state $|\psi\rangle$ is measured to have a position $\mathbf{x}$. But in relativistic QM (field theory), there may be a question as to how we measure position. **I CAN'T COME UP WITH ANY OBVIOUS REASON WHY IT SHOOULD BE MEANINGLESS TO MEASURE POSITION IN A RELATIVISTIC THEORY** but all questions of measurement need to be considered in terms of the interactions required, and Lorentz-invariance imposes some very strict constraints on what kinds of interactions, and therefore measurements, we can talk about.

- In the meantime, let's see how we could construct an appropriate observable. The point is that we can define any (self-adjoint) linear operator we want (and therefore an observable) by specifying some appropriate action on the basis vectors, in this case the vectors $|x\rangle$. Let's define the operator $\hat{Q}_z$ by

$$\hat{Q}_z|\mathbf{x}\rangle = z|\mathbf{x}\rangle. \tag{8}$$

  where as usual, z is the third component of the vector $\mathbf{x}$. This operator $\hat{Q}_z$ is diagonal with real values, and is therefore self-adjoint. It appears to be a position operator (measuring the z-component of the state).

- So now we have an operator – in fact, the only operator that would be a candidate for a position operator. But how do we know that it's appropriate to identify this operator $\hat{Q}_z$ with a measurement of the

[1] This is an abuse of notation. Different symbols should be used, for example $\tilde{\psi}$ and ψ to distinguish between the function and its Fourier transform.

z-coordinate of the state? As an example of how we'd answer this, consider how we could measure the position of a charged pion. We could aim a proton at it and then, because the pion is charged, its scattering behavior should depend on its location and its charge. So in an indirect way, the proton is involved in the measurement of the pion's position. If $\hat{Q}_z$ is a position operator (for the pion) then it must have some non-trivial relationship to the proton and in particular, to the charge of the proton.

- I don't think any of the above is especially helpful in deciding whether it's reasonable to call $\hat{Q}_z$ a position operator. But I believe that in the early days of non-relativistic QM, there were considerations of equal difficulty in coming to our current understanding of position observables etc.

- So, setting aside matters of interpretation, it appears we've found a way to define an x-wavefunction just like we did in nonrelativistic QM. The only apparent difference between relativistic and non-relativistic QM, is that in relativistic QM, we have $E = \sqrt{p^2 + m^2}$.

- If we also insist that the states should evolve as

$$\begin{aligned} |\psi, t\rangle &= e^{-i\hat{H}t}|\psi, 0\rangle \\ &= e^{-i\sqrt{\hat{p}^2+m^2}t}|\psi, 0\rangle \end{aligned} \tag{9}$$

then when we expand ψ as in eq. (5), we get

$$|\psi, t\rangle = \int d^3p e^{-i\sqrt{p^2+m^2}t}\psi(\mathbf{p})|\mathbf{p}\rangle. \tag{10}$$

Once all this has been stipulated, we can easily show that

$$\left(\partial^\mu\partial_\mu + m^2\right)\psi(\mathbf{x}, t) = 0. \tag{11}$$

In other words, the x-wavefunction satisfies the Klein-Gordon equation.

- Before turning to Thomson's treatment of this subject, consider the question of outside-the-light-cone-propagation which is worked out by Lancaster in his section 8.3.

  – Lancaster asks the question "if a particle starts off at position $\mathbf{x} = 0$ and then propagates for a time t, then what is its amplitude for being observed at a distance further than ct (i.e., that it traveled faster than the speed of light)?" I very temporarily introduced c instead of setting it to 1, so that it's clearer this describes faster-than-light travel. Now I'll return to natural units.

- Lancaster sets out to compute

$$\mathcal{A} = \langle \mathbf{x} | e^{-i\hat{H}t} | \mathbf{x} = 0 \rangle, \tag{12}$$

for the case $|x| > t$ (outside the light-cone).

- I'll leave it to you to follow the derivation, in particular the manipulations of Example 8.1. But the bottom line is that if we set $E = +\sqrt{p^2 + m^2}$, then $\mathcal{A}$ is non-zero outside the light cone. This is unacceptable since we can't allow the particle to have any amplitude for traveling faster than the speed of light.[2]

- What if the momentum eigenstates aren't a complete set? Let's hypothesize that momentum eigenstates come in two types $|\mathbf{p}, +\rangle$ and $|\mathbf{p}, -\rangle$. The two types are distinguished by the value of the Hamiltonian acting on those states:

$$\begin{aligned} \hat{H}|\mathbf{p}, +\rangle &= +\sqrt{p^2 + m^2}|\mathbf{p}, +\rangle \\ \hat{H}|\mathbf{p}, -\rangle &= -\sqrt{p^2 + m^2}|\mathbf{p}, -\rangle. \end{aligned} \tag{13}$$

In other words, the two types of momentum eigenstates correspond to positive and negative energies.

- Correspondingly, we'll need to extend our definition of $|\mathbf{x}\rangle$. I tried various things (remember, my goal is to get rid of the faster-than-light propagation). Define

$$\begin{aligned} |\mathbf{x}, 1\rangle &= \int d^3p e^{i\mathbf{p}\cdot\mathbf{x}} \left(|\mathbf{p}, +\rangle + i|\mathbf{p}, -\rangle \right) \\ |\mathbf{x}, 2\rangle &= \int d^3p e^{i\mathbf{p}\cdot\mathbf{x}} \left(|\mathbf{p}, +\rangle - i|\mathbf{p}, -\rangle \right) \end{aligned} \tag{14}$$

- Both of these states are reasonable candidates for a position eigenstate with eigenvalue $\mathbf{x}$. A new operator will need to be used to distinguish the two states but for now, we'll ignore that.

[2] It's a bit interesting to imagine how physicists from the 1920's would have reacted to this. After all, the theory of relativity was quite new and it must have occurred to them that in the quantum world, there might be violations of one sort and the other. As far as I know, the argument about propagation came later than the discoveries of Dirac and others.

- Consider the propagation question, now framed as the value of the amplitude

$$\mathcal{B} = \langle \mathbf{x}, 1|e^{-i\hat{H}t}|\mathbf{x} = 0, 1\rangle, \tag{15}$$

for $|x| > t$.

- To proceed, I'll need to dive into Lancaster's derivation of pages 75 and 76.
  - First insert a complete set of states ("resolution of the identity"), then apply the propagation operator to the momentum states, then expand $\langle \mathbf{x}, 1|$ as above, and finally use the orthonormality of the p-states.

$$\begin{aligned}
\mathcal{B} &= \langle \mathbf{x}, 1|e^{-i\hat{H}t}|\mathbf{x} = 0, 1\rangle \\
&= \int d^3p \left(\langle \mathbf{x}, 1|e^{-i\hat{H}t}|\mathbf{p}, +\rangle + i\langle \mathbf{x}, 1|e^{-i\hat{H}t}|\mathbf{p}, -\rangle\right) \\
&= \int d^3p \left(\langle \mathbf{x}, 1|\mathbf{p}, +\rangle e^{-iE(p)} + i\langle \mathbf{x}, 1|\mathbf{p}, -\rangle e^{+iE(p)}\right) \\
&= \int d^3p d^3p' e^{-i\mathbf{p}'\cdot\mathbf{x}}[\langle \mathbf{p}', +|\mathbf{p}, +\rangle e^{-iE(p)} - i\langle \mathbf{p}', -|\mathbf{p}, +\rangle e^{-iE(p)} + \\
&\qquad i\langle \mathbf{p}', +|\mathbf{p}, -\rangle e^{+iE(p)} + \langle \mathbf{p}', -|\mathbf{p}, -\rangle e^{+iE(p)}] \\
&= \int d^3p d^3p' e^{-i\mathbf{p}'\cdot\mathbf{x}}\delta(\mathbf{p}' - \mathbf{p})[e^{-iE(p)} + e^{+iE(p)}] \\
&= \int d^3p e^{-i\mathbf{p}\cdot\mathbf{x}}[e^{-iE(p)} + e^{+iE(p)}],
\end{aligned} \tag{16}$$

  where $E(p) = \sqrt{p^2 + m^2}$.
  - Up to a factor of $(2\pi)^{-3}$ this looks similar to Lancaster eq. (8.17) except that we are adding together one term with a positive energy and the other with a negative energy.
  - We can therefore follow the logic in Lancaster Example 8.1 in order to do the integral. This is a contour integral and requires an assessment of what contours to use. Lancaster explains his choice of contour by showing that in the upper semicircle, the factors die off exponentially. It turns out that precisely the same arguments show that even if the energy were negative, the factors die off on the upper semicircle. Therefore the only change between positive and negative energy is in the sign of t-exponents in the first line of eq.(8.19). That, copying Lancaster, leads to the conclusion –

up to constant factors

$$
\begin{aligned}
\mathcal{B} &= \int d^3p e^{-i\mathbf{p}\cdot\mathbf{x}}[e^{-iE(p)} + e^{+iE(p)}] \\
&= \frac{-i}{(2\pi)^3|\mathbf{x}|}\int_m^{+\infty} d(iz)ize^{-z|\mathbf{x}|}\left([e^{t\sqrt{z^2-m^2}} - e^{-t\sqrt{z^2-m^2}}] + [e^{-t\sqrt{z^2-m^2}} - e^{t\sqrt{z^2-m^2}}]\right) \\
&= 0.
\end{aligned} \tag{17}
$$

- So now we have the desired result. There is no propagation outside the light cone. It's easy to see that we'd get the same result if we replaced $|\mathbf{x}, 1\rangle$ by $|\mathbf{x}, 2\rangle$.

- **UNFORTUNATELY, WE GET A NONZERO AMPLITUDE FOR $\langle \mathbf{x}, 1|e^{-i\hat{H}t}|\mathbf{x} = 0, 2\rangle$.** That is, if we start with particle of type 2, then a portion of it can propagate faster than the speed of light and turn into a particle of type 1. (The same is true if we reverse 1 and 2.)

- Let's summarize. We started with a one-particle scalar theory with the ansatz that a complete set of states could be completely characterized by the value of the particle's 3-momentum. Furthermore, and in accordance with the fact that basis states are uniquely defined by their three-momentum we proposed that this theory only permitted positive energy eigenvalues. We then constructed another complete set of states that are candidates for position eigenstates. If we accept that interpretation, then we are led to the fact that there is a non-zero amplitude for those particles to propagate faster than the speed of light. This led us to propose that perhaps there is a degeneracy amongst momentum eigenstates. For each value of the 3-momentum, we allow the Hamiltonian operator to take on both a positive and a negative energy. We constructed a set of 'position' basis states. Once again there is a degeneracy. For every value of $\mathbf{x}$ there are two states that we call type-1 and type-2. If we examine the propagation of type-1 states into type-1 states, or type-2 states into type-2 states, the amplitude is zero for faster-than-light travel. That seems almost satisfactory. Unfortunately, there is a non-zero amplitude for faster-than-light mixing. So far, I have nothing to add to that (and no explanation apart from skepticism that there is really a relativistic version of a position operator).

- Now that we have allowed negative energies, we can meet up with Thomson's treatment of the Klein-Gordon equation. We have an x-wavefunction that satisfies the Klein-Gordon equation (this is true for both type-1 and type-2 particles and therefore for all linear combinations of them) so we can apply Thomson's analysis of the Klein-Gordon probability-current. For a plane wave Thomson obtains

$$\begin{aligned} \rho &= kE \\ \mathbf{j} &= k\mathbf{p} \end{aligned} \tag{18}$$

where k is a positive constant. But now that the energy E can be negative, this leads to a negative value of the probability density ρ. That's a problem!

There are a few other odds and ends related to this topic.

- I arrived at the existence of negative-energy states by showing that there is an issue with faster-than-light propagation. Thomson and others point out that negative-energy states are solutions of the Klein-Gordon equation, so the theory should allow them. Wikipedia (`https://en.wikipedia.org/wiki/Dirac_equation`) elaborates slightly on this, posing the issue in terms of an initial-value problem.

- The Wikipedia article also mentions that the second-order time derivative spoils the interpretation of $\psi(\mathbf{x})$ as a probability amplitude. This is, of course, the thrust of Thomson, Lancaster and others. In my notes above, I've puzzled over the question of how to interpret $\psi(\mathbf{x})$ if it isn't a probability amplitude – and whether that has to do with the meaning of a position observable.

- Everyone seems to agree that the right way out of this conundrum is to drop the non-relativistic notion of position wavefunctions as amplitudes, and instead introduce fields dependent on position. It's really a pretty dramatic re-casting of QM. What's surprising is that Heisenberg's algebraic formulation of QM, as well as the correspondence principle between Lagrangian variables and their conjugate momenta, managed to survive the transition from non-relativistic to relativistic QM.

- What is also surprising and fortuitous, is that the Dirac equation – which applies only to spin-$\frac{1}{2}$ particles – allows a Schrodinger-like interpretation of probability currents. As far as I can tell, this isn't a general feature of relativistic field theories, as we can see from the examination of the scalar particle. I found an article online which appears to go into much more depth on all this. The article is Probability in relativistic quantum mechanics and foliation of spacetime by Nicolic.

- I've looked at one more reference that is probably relevant. The paper is from 1963 and is quite significant: "Relativistic Invariance and Hamiltonian Theories of Interacting Particles" by Currie, Jordan and Sudarshan in Reviews of Modern Physics 35,Number 2 pp 350-375. The authors were setting out to describe a theory modelled on non-relativistic QM rather than field theory, but in which various observables are required to satisfy the Lorentz algebra. Their conclusion was surprising. On very general grounds, they showed that it wasn't possible to construct a theory of interacting particles. The best that could be done was to construct a theory of free particles. This paper probably would have received more notice, except that around 1963, an increasing number of physicists had become persuaded that field theory was the appropriate way to deal with relativistic QM, so they were no longer especially interested in the non-field-theoretic methods (such as what was being proposed in this paper). I don't think there is a direct connection between the Currie et al. paper, and the notes above concerning probability densities. However, there may be an indirect connection since the authors concern themselves at great length with the question of how to define position operators and, once defined, how these interplay with the geometry-generators (such as momentum) of the theory.

5 Connection with field theory

Up to now, I've stayed clear of the QFT formalism. But with a plausible modification of the previous analysis, I think there's an interesting connection to be made.

I'll jump to the standard expansion of a Klein-Gordon field in terms of creation and annihilation operators.

$$\phi(t,\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E(p)}\left(a_{\mathbf{p}}e^{-i(E(p)t-\mathbf{p}\cdot\mathbf{x})} + a^\dagger_{\mathbf{p}}e^{i(E(p)t-\mathbf{p}\cdot\mathbf{x})}\right). \tag{19}$$

Remember, $E(p)$ is defined to be $E(p) = \sqrt{p^2+m^2}$. Now let's apply that to the vacuum. The creation operator creates a state $|\mathbf{p}\rangle$ and the annihilation operator term is 0. So

$$\phi(t,\mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E(p)}e^{i(E(p)t-\mathbf{p}\cdot\mathbf{x})}|\mathbf{p}\rangle. \tag{20}$$

The integral can be shown to be Lorentz invariant by noting that for any function f,

$$\int \frac{d^3p}{\sqrt{2E(|\mathbf{p}|)}} f(E(|\mathbf{p}|), \mathbf{p}) = \int d^4p f(p)\delta(p^2 - m^2) \tag{21}$$

where on the right, p refers to the 4-vector. So, for example, on the right we have $p^2 = p_0^2 - |\mathbf{p}|^2$. If we set $t = 0$, then eq.(20) becomes

$$\phi(0, \mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E(p)}} e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle, \tag{22}$$

which is similar to eq. (6), but with the integration measure replaced by a Lorentz-appropriate measure. This suggests that perhaps a better definition of the state $|\mathbf{x}\rangle$ would be the left side of the above equation. So, let's define a new 'Lorentz-covariant' state

$$|\mathbf{x}, t\rangle_L = \phi(t, \mathbf{x})|0 > . \tag{23}$$

This seems to be as valid a candidate for a position-state (or even more valid) as the previously defined position-state. Then we have

$${}_L\langle \mathbf{x}', t|\mathbf{x}, t\rangle_L = \langle 0|\phi(t, \mathbf{x}')\phi(t, \mathbf{x})|0\rangle. \tag{24}$$

We've invoked the fact that $\phi^* = \phi$. The right-hand side is called the propagator and is **not** a position delta-function. So these position-states don't form an orthonormal set and thus don't form an appropriate basis from which to develop a position-wavefunction that can be interpreted as a probability amplitude for determining position.

Even so, we can define a quantity $\psi_L(\mathbf{x}, t) = {}_L\langle \mathbf{x}, t|\psi\rangle$ and this 'feels' like it is a local measurement – using the field $\phi(t, \mathbf{x})$ – of the state $|\psi\rangle$.

It turns out that the field-theoretic causality condition (information can't travel faster than the speed of light) is

$$[\phi(x'), \phi(x)] = 0 \tag{25}$$

if the 4-vectors x and x' are outside one another's light cones. This follows from the fact that commuting operators can be measured simultaneously, and therefore one cannot affect the other. This has the consequence that

$$\text{Im } {}_L\langle \mathbf{x}', t'|\mathbf{x}, t\rangle_L = 0 \tag{26}$$

if $|x - x'| > |t - t'|$.