Thomson Chapter 4 – Dirac equation

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1 Thomson 4.1 and 4.2: The free electron's Hamiltonian

If we want to compute scattering cross-sections (aka transitions) for electron collisions, we need a Hamiltonian of the form $H_0 + H_I$. Thomson Chapter 4 examines the electron Hilbert space and H_0, describing free motion of electrons. The resulting equation of motion is the *Dirac equation*.

1.1 Thomson 4.1: A wrong guess – the Klein-Gordon equation

Several things go wrong when we modify Schrodinger's equation for relativity. Remember

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} \tag{1}$$

In a Lorentz-invariant theory, the time-component must behave just like the space-component. So if we have the second derivative with respect to space, we must have the second derivative with respect to time.

Remember also, that Schrodinger identified the energy operator as $\hat{E} = i\frac{\partial}{\partial t}$ and the momentum operator as $\hat{\vec{p}} = -i\nabla$. With these identifications, eq. (1) is equivalent to

$$\hat{E} = \frac{\hat{\mathbf{p}}^2}{2m} \tag{2}$$

which is a non-relativistic equation. The corresponding relativistic (and Lorentz-invariant) equation would be

$$\hat{E}^2 = \hat{\mathbf{p}}^2 + m^2. \tag{3}$$

If we now rewrite the operators we get the wave equation

$$\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi = -m^2\psi \tag{4}$$

which can be written in tensor notation as

$$\left(\partial^\mu \partial_\mu + m^2\right)\psi = 0. \tag{5}$$

This is the Klein-Gordon equation.

1.1.1 Issue 1 – negative energies

A basis of solutions to eq. 5 is

$$\psi_{\mathbf{p}}(t, \mathbf{x}) = N_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x} - Et)} \tag{6}$$

where

$$E = \pm\sqrt{\mathbf{p}^2 + m^2} \tag{7}$$

The problem is that some energy values are arbitrarily negative.

It's not necessarily obvious this is a problem. For many classical theories, this kind of thing ultimately leads to instabilities. Dirac and others had that experience.

1.1.2 Issue 2 – insufficiency of single-particle Hilbert space

The Klein-Gordon equation looks like a simple relativistic extension of the Schrodinger equation. So we might hope that the wave-function can be interpreted in a similar way, and that an abstract formulation of a single-particle wavefunction, is that it is part of a single-particle Hilbert space.

Heisenberg's uncertainty principle applies (a consequence of Fourier transforms). Namely, $\Delta p \Delta x > \hbar/2$. So if the electron were confined to a small enough space (for example, something like the 'classical radius' of the electron) then its momentum would be so uncertain that there would be a finite probability that $p^2 >> m$. Since $E = \sqrt{\mathbf{p}^2 + m^2}$, this would mean $E >> m$ and therefore that enough energy would be available to create multiple electrons.

This has the apparent consequence that a one-particle Hilbert space is inadequate to describe a system that begins as one particle, since if the system is sufficiently confined, it could turn into multiple particles.

This is not necessarily a problem. Just because it's kinematically possible (i.e., allowed by energy and momentum conservation) for one particle to transform to many, there needn't be a process that enables such a transition.

Still, the kinematic considerations apply to any and all particles, and it would be surprising if all of them were prohibited from transforming into multiple particles.

1.1.3 Issue 3 – propagation outside the light cone

This topic is covered in section 8.3 of Lancaster (with details and a nice illustration). Recall that the exponential of the energy operator – sometimes known as the Hamiltonian – acts on states by propagating them in time. Namely,

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \tag{8}$$

where $H \equiv \hat{E}$. Suppose that the particle's starting state is an eigenstate of the position operator, with eigenvalue $\mathbf{x} = 0$. What is the probability that after a time t, the state can be found at position x? In other words, what is $|\langle x|e^{-iHt}|0\rangle|^2$?

We can compute that by using the basis of eqs. (6) and (7). Lancaster does this computation. What we discover is that even if the particle is unable to classically reach position x in time t – owing to the limitation of the speed of light, in quantum mechanics **there is a non-zero probability of finding the particle at position** x.

This particular issue can't be dismissed by any simple argument.

1.2 A proposal for addressing Klein-Gordon issues

Historically, the first issue – negative energies – was the one that Dirac set out to address. He was led to a new equation. It still had a negative-energy problem. So he then tackled that problem, but in the context of the new equation.

This led to the discovery of positrons, or more generally, antiparticles. Many textbook treatments follow this history. However, it's mathematically unclean and highly limited. Dirac himself resolved issues by inventing quantum field theory. In my opinion, that's how the subject should be taught to purists.

Thomson follows the traditional historical approach. Even though it is on shaky ground from a foundational perspective, the historical approach makes the Dirac theory a reasonably straightforward generalization of the non-relativistic Schrodinger theory. So I'll follow Thomson. First, a bridge section somewhat parallel to Lancaster's approach.

To set up the scattering theory, we need to begin with a Hilbert space, and a free Hamiltonian both of which conspire to address the Klein-Gordon issues.

1.2.1 A Hilbert space for electrons and positrons

- A simple non-relativistic electron has spin-$\frac{1}{2}$, i.e., *two components* that transform amongst one another when we rotate reference frames.
- Dirac predicted that the relativistic theory has *four components*, interpreted as 2 electron spins and 2 positron spins.
- So the Hilbert space has a basis of single-particle states characterized as

$$|\mathbf{p}, e^{\pm}, s\rangle, \tag{9}$$

  where $e^{\pm}$ takes the value 1 or -1 depending on whether we have a positron (e^{+}) or electron (e^{-}) ; 's' is $\frac{1}{2}$ or $-\frac{1}{2}$. The above state is an eigenstate of the 3-momentum operator $\hat{p}_k$ (not yet specified) whose eigenvalues are p_k.
- The Hilbert space also has multi-particle states. For example, a two-particle state is $|\mathbf{p}, e^{\pm}, s, \mathbf{p}', e'^{\pm}, s'\rangle = -|\mathbf{p}', e'^{\pm}, s', \mathbf{p}, e^{\pm}, s\rangle$. Notice that the order matters.

1.2.2 Observables – momentum, energy, position

- Except when absolutely necessary, I'll suppress the time variable for states. Just append t if you need it.
- If we defined the Hilbert space in terms of eigenfunctions of momentum ($|\mathbf{p}, e^{\pm}, s\rangle$) then the momentum operator is defined as the (vector) operator which acts on states as

$$\hat{p}_k|\mathbf{p}, e^{\pm}, s\rangle = p_k|\mathbf{p}, e^{\pm}, s\rangle. \tag{10}$$

  This is really just a definition. But it's the starting point for defining the position operator and position eigenfunctions. First, what about energy?
- The energy operator $\hat{H}$ or $\hat{E}$ is defined by

$$\hat{H}|\mathbf{p}, e^{\pm}, s\rangle = \sqrt{p^2 + m^2}|\mathbf{p}, e^{\pm}, s\rangle. \tag{11}$$

  Notice that the energy eigenvalue is always positive. The Hamiltonian is also responsible for time evolution (now we need to unsuppress the time-variable).

$$|\psi, t\rangle = e^{-i\hat{H}t}|\psi, 0\rangle. \tag{12}$$

- Next, the question is "what is the position (vector) operator?" I'll define that indirectly in terms of the position eigenfunctions $|\hat{x}, e^{\pm}, j\rangle$ where j ranges from 1 to 4.

$$\begin{aligned} \langle \mathbf{x}, e^-, j| &= \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \langle \mathbf{p}, e^-, s|(u_s(\mathbf{p}))_j \\ \langle \mathbf{x}, e^+, j| &= \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \langle \mathbf{p}, e^+, s|(v_s^*(\mathbf{p}))_j \end{aligned} \tag{13}$$

  and the 4-vectors (known as *spinors*) are given in Thomson by eqs. (4.51) and (4.52). For example

$$u_2(\mathbf{p}) = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix} \tag{14}$$

  Notice that there are twice as many x-type states as p-type states, so half of these are linearly dependent on the others.

  If you are bothered by this abstract sum of vectors, remember that we are generally interested in the inner products, representing a probability amplitude. So, for example,

$$\langle \mathbf{x}, e^-, j|\psi\rangle = \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \langle \mathbf{p}, e^-, s|\psi\rangle (u_s(\mathbf{p}))_j \tag{15}$$

  Now we have a bona fide Fourier transform. The inner product $\langle \mathbf{x}, e^-, j|\psi\rangle$ is the probability amplitude that the state $|\psi\rangle$ has a 'component' $|\mathbf{x}, e^-, j\rangle$. What does that mean? It means this. Suppose I have prepared, in some way, the state $|\psi\rangle$. Then I perform a measurement whose outcome happens to be the state $|\mathbf{x}, e^-, j\rangle$. How do we know that this is the outcome? In principal, we know because we did a certain set of measurements and observed a certain set of numbers. But it's nontrivial to make the connection between certain observations and the resultant states. This is called "associating an observation/measurement with an operator on the Hilbert space – often known as an 'observable'.

- The position operator is then defined as the (vector) operator which acts on states as

$$\hat{x}_k|\mathbf{x}, e^{\pm}, j\rangle = x_k|\mathbf{x}, e^{\pm}, j\rangle. \tag{16}$$

- Why is $\hat{x}$ interpreted as the position operator? Another way of asking that question is whether there is a position observable (a measurement) that can be represented as the operator $\hat{x}$. That leads to asking what is the ultimate meaning of momentum and position? Momentum is defined in Hamiltonian and Lagrangian mechanics, as the variable 'conjugate' to position – a member of the cotangent space to configuration space. In quantum theory, the momentum operator is the differential translation operator.

  Momentum and position operations are related to one another via Fourier transforms. But beyond that observation, remain some tricky questions that I don't understand well enough to comment intelligently about.

  More generally, one could ask why the form of the position eigenfunctions in eq. (13) is so complicated. As in the previous paragraph, the answer lies in the necessity of having various operators reflect the Lorentz geometry of the system. In particular, there are relationships required between the operators for translations and rotations – with corresponding relationships between operators for 4-momentum (energy plus 3-momentum) and angular momentum. We haven't discussed rotations or angular momentum, but these are responsible for most of the messiness of spinor theories.

1.2.3 Dirac wavefunctions

Now that we've defined the position operator and position eigenfunctions, we're ready to look at the relativistic version of the Schrodinger equation. Remember that in non-relativistic QM, the probability that a state $|\psi\rangle$ has position $\mathbf{x}$ is given by $|\langle \mathbf{x}|\psi\rangle|^2$ where $\langle \mathbf{x}|$ is the eigenfunction of $\hat{\mathbf{x}}$.

In the relativistic theory, there is at least one minor complication. There are multiple eigenfunctions of $\hat{\mathbf{x}}$, depending on whether we are looking at an electron or positron, and also depending on some combination of spins. So we have to be more precise about what measurements we're making. I don't know whether the Dirac wavefunctions resolve the issue, noted in the Klein-Gordon theory, of particles propagating faster than the speed of light. This calculation could be done, but I think it's messy. I have a feeling we will still have the same problem.

Now let's see how we resolve the negative-energy issue. Define the prob-

ability amplitude for the time-evolved electron momentum eigenfunction.

$$\begin{aligned}\psi_s^{\mathbf{p}}(\mathbf{x},t)_j &= \langle \mathbf{x}, e^-, j, 0 | \mathbf{p}, e^-, s, t\rangle \\ &= \langle \mathbf{x}, e^-, j, 0 | e^{-i\hat{H}t} | \mathbf{p}, e^-, s, 0\rangle \\ &= \langle \mathbf{x}, e^-, j, 0 | e^{-i\sqrt{p^2+m^2}t} | \mathbf{p}, e^-, s, 0\rangle \\ &= \frac{1}{(2\pi)^3} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})} u_s(\mathbf{p})_j\end{aligned} \tag{17}$$

where $E = \sqrt{p^2+m^2}$.

This function, generally known as the Dirac plane-wave solution for electrons, satisfies the (Dirac) equation

$$(\gamma^\mu \hat{p}_\mu - m)\, \psi_s^{\mathbf{p}}(\mathbf{x},t) = 0, \tag{18}$$

or in shorthand notation

$$(\not{\hat{p}} - m)\, \psi_s^{\mathbf{p}}(\mathbf{x},t) = 0. \tag{19}$$

The gamma matrices are given in Thomson and $\hat{\mathbf{p}} = -i\nabla$.

Now define the probability amplitude for the time-evolved positron momentum eigenfunction.

$$\begin{aligned}\tilde{\psi}_s^{*\mathbf{p}}(\mathbf{x},t)_j &= \langle \mathbf{x}, e^+, j, 0 | \mathbf{p}, e^-, s, t\rangle \\ &= \langle \mathbf{x}, e^-, j, 0 | e^{-i\hat{H}t} | \mathbf{p}, e^-, s, 0\rangle \\ &= \langle \mathbf{x}, e^-, j, 0 | e^{-i\sqrt{p^2+m^2}t} | \mathbf{p}, e^-, s, 0\rangle \\ &= \frac{1}{(2\pi)^3} e^{-i(Et-\mathbf{p}\cdot\mathbf{x})} v_s^*(\mathbf{p})_j.\end{aligned} \tag{20}$$

Observe that we defined the conjugate function $\tilde{\psi}_s^{*\mathbf{p}}(\mathbf{x},t)$ which is also a valid probability amplitude (the probability is the conjugate-squared).

$\tilde{\psi}$, known as the Dirac plane-wave solution for positrons, satisfies the (Dirac) equation

$$(\not{\hat{p}} + m)\, \tilde{\psi}_s^{\ \mathbf{p}}(\mathbf{x},t) = 0. \tag{21}$$

Note that $\tilde{\psi}_s^{\mathbf{p}}(\mathbf{x},t) = \frac{1}{(2\pi)^3} e^{i(Et-\mathbf{p}\cdot\mathbf{x})} v_s(\mathbf{p})$.

WE SEE THAT THE POSITRON PLANE-WAVE SOLUTION HAS 'NEGATIVE ENERGY'. THAT IS, THE COEFFICIENT OF ENERGY IN THE EXPONENT IS $+it$ RATHER THAN $-it$. But by construction, the states were all positive-energy so there is no negative-energy issue.

2 Spin

The subject of rotations, spin and angular momentum – for Dirac particles – has been treated several times in the past couple of years. It's worth doing again so we can connect with Thomson.

- The 'spin' of an electron, is regarded as a property of some kind of internal degrees of freedom of the electron – much as if the electron were a solid object like a top. This is just a mnemonic.

- Angular momentum is, classically, proportional to 'rate of rotation'. For example, if a top rotates around the z-axis, then its angular momentum is

$$L_{\text{top}} = I\omega \tag{22}$$

  where $\omega = \frac{d\theta}{dt}$ where θ is the angle of rotation around the z-axis.

- Thomson, page 104, tells us that the spin operator for Dirac wavefunctions is

$$\hat{S}_z = \frac{1}{2}\Sigma_z \equiv \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{23}$$

  This expression is using the Pauli-Dirac representation of the gamma matrices.[1]

- Rotations by an angle θ are expressed as [2]

$$\hat{R}_z(\theta) = e^{i\hat{S}_z\theta} \tag{24}$$

  so if we have a Dirac wavefunction $\psi_i(\mathbf{0})$, the rotation acts as

$$\hat{R}_z(\theta)\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} e^{i\frac{\theta}{2}}\psi_1 \\ e^{-i\frac{\theta}{2}}\psi_2 \\ e^{i\frac{\theta}{2}}\psi_3 \\ e^{-i\frac{\theta}{2}}\psi_4 \end{pmatrix}. \tag{25}$$

  I chose the coordinates to be $\mathbf{0}$ because I didn't want, yet, to deal with how those rotate. If you look, example, at the notes Introduction to the Dirac Equation, you'll see how to rotate coordinates around the z-axis, and then you'll see how to demonstrate that the Dirac equation is invariant when you rotate both the indices and the coordinates.

[1] In previous notes I tended to prefer the Weyl representation so you may see differences.
[2] I'm being a bit careless with the sign in the exponent. There might be a minus sign.

- Recall the Dirac wavefunctions defined in eq. (17), $\psi_s^{\mathbf{P}}(\mathbf{x}, t)$. These are proportional to $u_s(\mathbf{p})$. Thomson shows, on page 104, that

$$\hat{S}_z u_s(\mathbf{p}) = s u_s(\mathbf{p}) \tag{26}$$

  where $s = \pm\frac{1}{2}$ and $\mathbf{p} = \pm(0, 0, 0, p)$.

3 Homework

- **EXERCISE** Make sure you follow the section of Thomson at the beginning of section 4.8 in which he derives eq. (26) and then show that the states $\psi_s^{\mathbf{P}}(\mathbf{x}, t)$ are eigenvectors of the spin operator. (Take $\mathbf{p}$ to be in the z-direction as above.) Find the eigenvalues.

- Read sections 4.8 and 4.9, especially about helicity. I'll cover this briefly, but it's an important aspect of how people think about electrons and other spin $\frac{1}{2}$ particles. Gather questions to ask.

- **SOLUTION**

  First apply the spin operator to the spinors $u_s(\mathbf{p})_j$ which appear in eq. (17) above. Take, for example, u_1 in Thomson on page 104 (but set the normalization factor N to 1 – this just makes the notation easier and doesn't change the final answer) and $\hat{S}_z$ from eq. (23)

$$\begin{aligned} \hat{S}_z u_1(\mathbf{p}) &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{\pm p}{E+m} \\ 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ \frac{\pm p}{E+m} \\ 0 \end{pmatrix} \\ &= \frac{1}{2} u_1(\mathbf{p}). \end{aligned} \tag{27}$$

where $E = \sqrt{p^2 + m^2}$. Similarly

$$\begin{aligned}\hat{S}_z u_2(\mathbf{p}) &= \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\\ \frac{\pm p}{E+m}\end{pmatrix}\\ &= -\frac{1}{2}\begin{pmatrix}0\\ 1\\ 0\\ \frac{\pm p}{E+m}\end{pmatrix}\\ &= -\frac{1}{2}u_2(\mathbf{p}).\end{aligned} \tag{28}$$

Then, to solve the problem, recall from eq. (17) that

$$\psi_s^{\mathbf{p}}(\mathbf{x}, t)_j = \frac{1}{(2\pi)^3} e^{-i(Et - \mathbf{p}\cdot\mathbf{x})} u_s(\mathbf{p})_j \tag{29}$$

Now apply the spin operator above, noting that it acts only on the spinors $u_s(\mathbf{p})$. For example,

$$\begin{aligned}\hat{S}_z \psi_2^{\mathbf{p}}(\mathbf{x}, t) &= \frac{1}{(2\pi)^3} e^{-i(Et - \mathbf{p}\cdot\mathbf{x})} \hat{S}_z u_2(\mathbf{p})\\ &= -\frac{1}{2}\psi_2^{\mathbf{p}}(\mathbf{x}, t).\end{aligned} \tag{30}$$

4 Helicity

The subject of helicity is covered in Thomson section 4.8.1. In my opinion, there is way too much detail here for our purposes.

Here's the definition:

$$\hat{\mathbf{h}} \equiv Helicity = \frac{\hat{\mathbf{S}} \cdot \mathbf{p}}{p} \tag{31}$$

where $\hat{\mathbf{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ and $\mathbf{p}$ is the 3-momentum for a general direction (no longer the z-direction). The spin operators here are specifically defined for the spinor representation (of spin-$\frac{1}{2}$) but the concept can easily be generalized for other kinds of particles like photons or gravitons (scalars are boring since they have no spin).

When the momentum is in the z-direction, helicity is proportional to the z-spin, so this can simply be regarded as 'spin in the motion of direction'.

Thomson gives two reasons why helicity is a concept that replaces spin in many applications:

- The Dirac spinors above are eigenstates of the spin-z operator. However, that's only true when the momentum is in the z-direction. So if the electron is traveling in some other direction, its Dirac spinors aren't eigenvectors of the spin-z operator and therefore aren't necessarily as 'interesting' or easy to work with.

- As a related matter, the spin operator generally doesn't commute with the Dirac Hamiltonian (except when it is operating on plane waves moving in the z-direction). One consequence is that energy eigenstates – which correspond to lines in the spectrum – don't necessarily correspond to spin eigenstates.

Frankly, I don't regard either of these two explanations as especially compelling reasons to concern us.

I checked what Schwartz has said by way of motivation and he seems less interested in helicity (other than for completeness of presentation). There is yet a different concept called chirality which I'll skip for now. That matters more since it is very relevant in weak interactions. For massless or very light or very slow particles, chirality and helicity are effectively the same.

Lancaster seems to have the same view as Schwartz.

Historically, the notion of helicity is used when describing photons, in which case it is known as circular polarization.

Finally, Thomson introduces helicity spinors in his equation 4.66 and 4.67. These are analogues of the Dirac spinors he introduced earlier, but with the property that they are eigenvectors of the helicity.

5 Take-aways

What can we do with all this Dirac technology?

- **Spectroscopy**. The Dirac equation is like the Schrodinger equation. We can solve the eigenvalue equation. What's missing so far, is the 'potential' function. We've only looked at free particles and their spectrum is boring (plane waves with continuous eigenvalues, or discrete if the electrons are in a box).

  We can add a potential (electromagnetic) and then find the eigenvalues of, for example, the hydrogen atom. *By the way, we don't need to interpret the meaning of the Dirac wavefunction ψ which appears in the Dirac equation. If all we care about is the spectrum, then that wavefunction may or may not have anything to do with probability or field operators etc. It just doesn't matter.*

- **Scattering**. The tool we use is perturbation theory, represented schematically by Feynman diagrams. This is a bunch of rules derived from perturbative field theory, where the leading terms correspond to free non-interacting particles. The general approach starts with particle and antiparticle states of the form $\langle \mathbf{x}, e^{\pm}, j|$ discussed above in eq. (13) and which involve the Dirac spinors $u_s(\mathbf{p}$ etc. Once again, we don't need to interpret ψ.

6 Feynman diagram-ology

6.1 Perturbation Theory – copied from notes on Thomson Chapter 1

The mathematical interpretation of Feynman diagrams is in the context of the calculation of either decay amplitudes or scattering amplitudes. If an amplitude is denoted A, then the probability is denoted $|A|^2$. Feynman diagrams are schematics for calculations of amplitudes.

A decay amplitude (per unit time) is the amplitude that a single particle decays into a particular set of outgoing particles. Also of interest is the cumulative decay amplitude for the particle to decay into anything. This quantity is inversely proportional to the particle's lifetime.

A scattering amplitude is the amplitude that two colliding particles produce a particular set of outgoing particles.

- Amplitudes are derived from path integrals by computing moments of the form

$$G^{i_1 \ldots i_n}(y_1, ..., y_n) \equiv \int \mathcal{D}\phi^i e^{i(\int d^4x \mathcal{L}(\phi^i))(x)+i\epsilon)} \prod_{j=1}^{n} \phi^{i_j}(y_j). \qquad (32)$$

  In this expression, the Lagrangian is a function of fields shown generically as ϕ but which can represent, for example A^μ or ψ^{ν_e}.

- If the Lagrangian were quadratic in the fields, then these moments could be computed exactly, in the same way that moments of a Gaussian distribution can be computed exactly.

- This enables us to compute perturbatively. Rewrite the Lagrangian generically as

$$\mathcal{L} = \mathcal{L}_0 + \lambda \mathcal{L}_I \qquad (33)$$

where $\mathcal{L}_0$ is the part of the Lagrangian which is quadratic (this part is called the free Lagrangian) and $\lambda\mathcal{L}_I$ is the rest, and is called the interaction term. Here λ is a coupling constant which, for the purposes of perturbation theory, must be small. An example of the electromagnetic interaction term is:

$$\begin{aligned}\mathcal{L}_{EM} &= -e\bar{\psi}^e \not{A}\psi^e \\ &\equiv -e\sum_{i,j,k,\mu}\psi_i^{e\dagger}\gamma^0_{ij}\gamma^\mu_{jk}A_\mu\psi^e_k + ...\end{aligned} \tag{34}$$

- Then we can expand the exponential in the path integral as

$$\begin{aligned}&e^{i\int d^4x(\mathcal{L}_0+\lambda\mathcal{L}_I+i\epsilon)} = e^{i\int d^4x(\mathcal{L}_0+i\epsilon)}e^{i\lambda\int d^4x\mathcal{L}_I} \\ &= e^{i\int d^4x(\mathcal{L}_0+i\epsilon)}\left(1+i\lambda\int d^4y_1\mathcal{L}_I + i^2\frac{\lambda^2}{2!}\int d^4y_1\mathcal{L}_I\int d^4y_2\mathcal{L}_I + ...\right)\end{aligned} \tag{35}$$

- Now, when we compute the path-integral, the exponential term is quadratic, and the interaction piece has become a series contributing to the moments – all of which are calculable.

- Consider the term $i^2\frac{\lambda^2}{2!}\int d^4y_1\mathcal{L}_I\int d^4y_2\mathcal{L}_I$ where, for example, we take $\lambda\mathcal{L}_I$ to be the electromagnetic term above. This becomes[3]

$$-e^2\int d^4y_1d^4y_2\left(\bar{\psi}^e\not{A}\psi^e\right)(y_1)\left(\bar{\psi}^e\not{A}\psi^e\right)(y_2) \tag{36}$$

- **This represents the schematic of Fig 1.5 from Thomson**.

[3]For sticklers, please note that we haven't ever said what we mean by path integrals involving fermions. Technically, these fields are regarded as members of a Grassman algebra, and integration is defined rather differently than it would be for regular complex-valued fields. However, miraculously, the formalism works with few modifications.

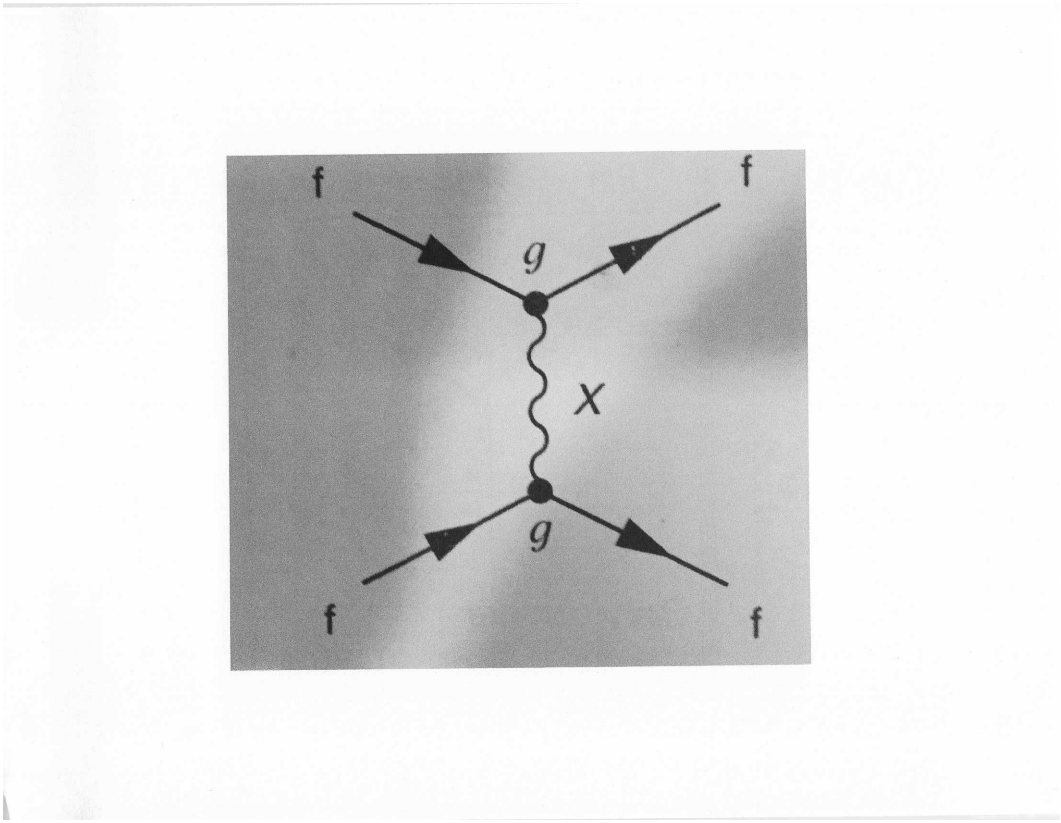


Figure 1: Thomson Figure 1.5

The vertex appears twice. It turns out that we don't have to path-integrate the fields which appear as free particles (i.e., altogether 4 fermion fields). The moment-integral only has to do with the two appearances of A. This is the meaning of the connected line. The moment-integral corresponding to the connected line is known as the propagator and has the generic form $\frac{\text{stuff}}{p^2-m^2+i\epsilon}$. The momentum appearing in this expression is the momentum mentioned in the previous section and which, as mentioned in the previous section does **not** have the property that $p^2 = m^2$.[4]

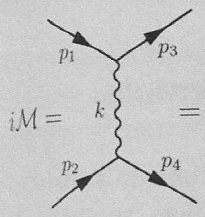
[4]Since gluons are massless, this particular example would have $m^2 = 0$.

6.2 Scattering rules – Dirac spinors

tion, but it will also [illegible] corrections. In this whole section, we neglect any internal structure of the proton, treating it, like the muon, as a pointlike particle. A discussion of what actually happens at extremely high energy, $E_{\rm CM} \gg m_p$, is given in Chapter 32.

13.4.1 QED amplitude

As far as QED is concerned, a proton and a muon are the same thing, up to the sign of the charge, which gets squared anyway, and the mass. So let us start with $e^-\mu^- \to e^-\mu^-$. The amplitude is given by a t-channel diagram:



$$i\mathcal{M} = (-ie)\bar{u}(p_3)\gamma^\mu u(p_1)\frac{-i\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]}{(p_1 - p_3)^2}(-ie)\bar{u}(p_4)\gamma^\nu u(p_2), \tag{13.82}$$

with $k^\mu = p_1^\mu - p_3^\mu$. As in $e^+e^- \to \mu^+\mu^-$, the $k^\mu k^\nu$ term drops out for on-shell spinors, as expected by gauge invariance. So this matrix element simplifies to

$$\mathcal{M} = \frac{e^2}{t}\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2), \tag{13.83}$$

with $t = (p_1 - p_3)^2$. Summing over final states and averaging over initial states,

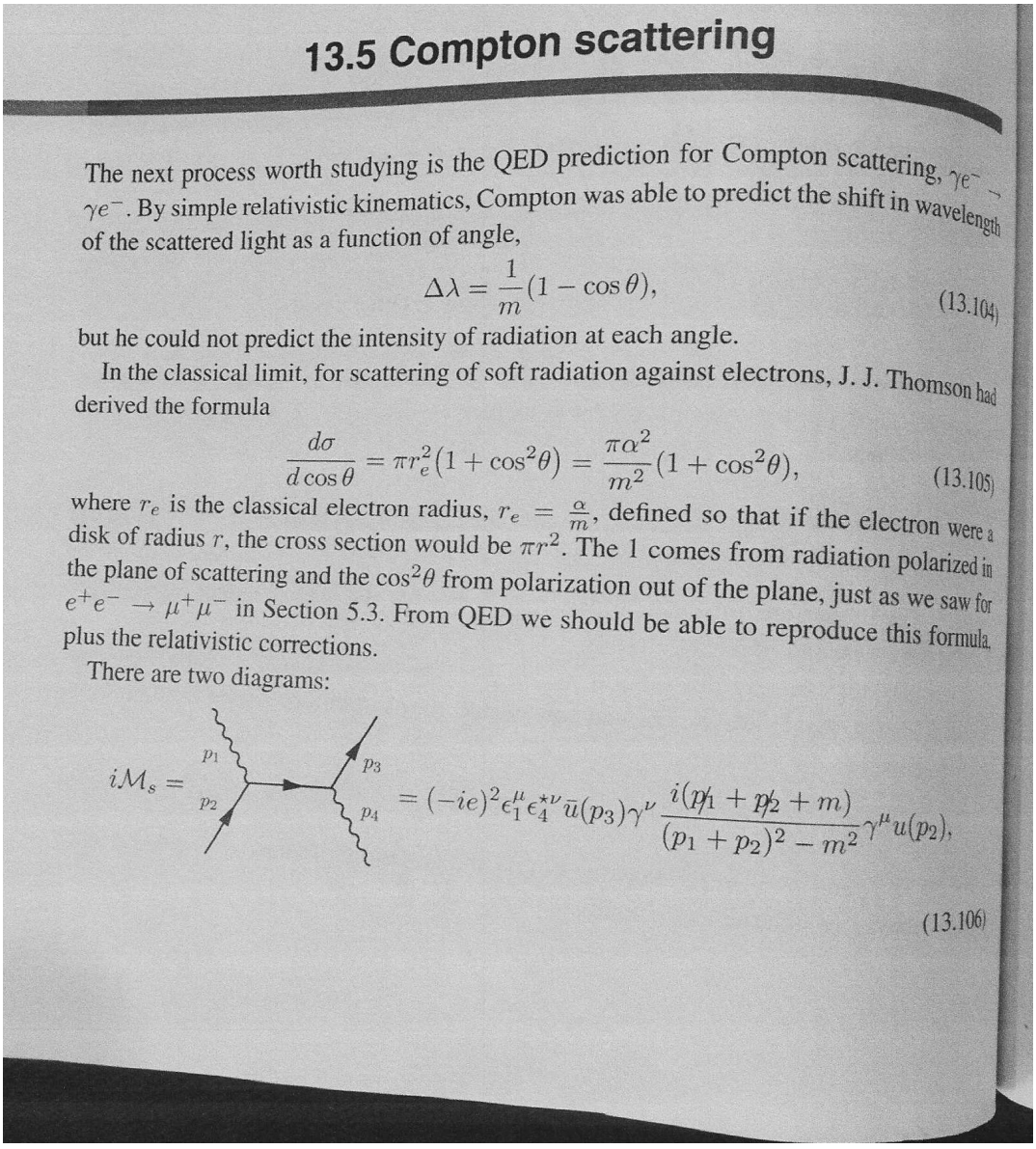
$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4t^2}\mathrm{Tr}[(\not{p}_1 + m_e)\gamma_\nu(\not{p}_3 + m_e)\gamma^\mu]\mathrm{Tr}[(\not{p}_4 + m_\mu)\gamma_\mu(\not{p}_2 + m_\mu)\gamma^\nu]. \tag{13.84}$$

Figure 2: Rutherford Scattering(from Schwartz))

In this image we fill in some of the details of the perturbation theory. Corresponding to each of the free spin-$\frac{1}{2}$ particles (two going in and two going out) you have to multiply spinors corresponding to each particle. For example, you have the term $\bar{u}(p_3)\gamma^\mu u(p_1)$ which is shorthand notation for $u^T(p_3)\gamma_0\gamma^\mu u(p_1)$. [5] You'll see, if you look at eq. (34), that this product resembles the interaction term, where the Dirac spinors are in the place of the fields ψ. You might wonder what happens to the A_μ appearing in eq. (34). This is the electromagnetic field and shows up in the diagram as a constituent of the photon propagator – the wavy line which is represented by the denominator of equation 13.82.

[5] Schwartz uses a common convention where momenta written in boldface refer to the three spacial components of the momentum, and if the momentum is written in regular font, it represents the four components of momenta. When dealing with free particles, you have a dispersion relation which relates the 0 component to the others. That dispersion relation cannot be used for the joined lines in diagrams.

6.3 Scattering rules – Dirac propagator



13.5 Compton scattering

The next process worth studying is the QED prediction for Compton scattering, $\gamma e^- \to \gamma e^-$. By simple relativistic kinematics, Compton was able to predict the shift in wavelength of the scattered light as a function of angle,

$$\Delta\lambda = \frac{1}{m}(1 - \cos\theta), \tag{13.104}$$

but he could not predict the intensity of radiation at each angle.

In the classical limit, for scattering of soft radiation against electrons, J. J. Thomson had derived the formula

$$\frac{d\sigma}{d\cos\theta} = \pi r_e^2(1 + \cos^2\theta) = \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta), \tag{13.105}$$

where r_e is the classical electron radius, $r_e = \frac{\alpha}{m}$, defined so that if the electron were a disk of radius r, the cross section would be πr^2. The 1 comes from radiation polarized in the plane of scattering and the $\cos^2\theta$ from polarization out of the plane, just as we saw for $e^+e^- \to \mu^+\mu^-$ in Section 5.3. From QED we should be able to reproduce this formula, plus the relativistic corrections.

There are two diagrams:

$$i\mathcal{M}_s = \quad p_1 \quad p_2 \quad p_3 \quad p_4 \quad = (-ie)^2\epsilon_1^\mu\epsilon_4^{\star\nu}\bar{u}(p_3)\gamma^\nu\frac{i(\not{p}_1 + \not{p}_2 + m)}{(p_1 + p_2)^2 - m^2}\gamma^\mu u(p_2), \tag{13.106}$$

Figure 3: Compton Scattering(from Schwartz))

The main point of showing this image, is that the 'joined' line here, isn't wavy. In fact, it is the fermion propagator. It joins two vertices and, not surprisingly, must contract one of the spinors from each of the two joined

vertices. We end up with a sum over spins:

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not{p} - m. \tag{37}$$

You'll notice this in the numerator of equation 13.106 (the momentum of the middle fermion line is $p_1 + p_2$).

7 Afterwords

Some issues surfaced during the writing and presentation of the above notes. I have a separate document that attempts (and partly fails) to resolve those issues. See `https://billcelmaster.com/wp-content/uploads/2022/01/Some-basic-relativistic-quantum-mechanics-puzzles.pdf`