# Making sense of QFT

## Lecture 2: Wigner's particles

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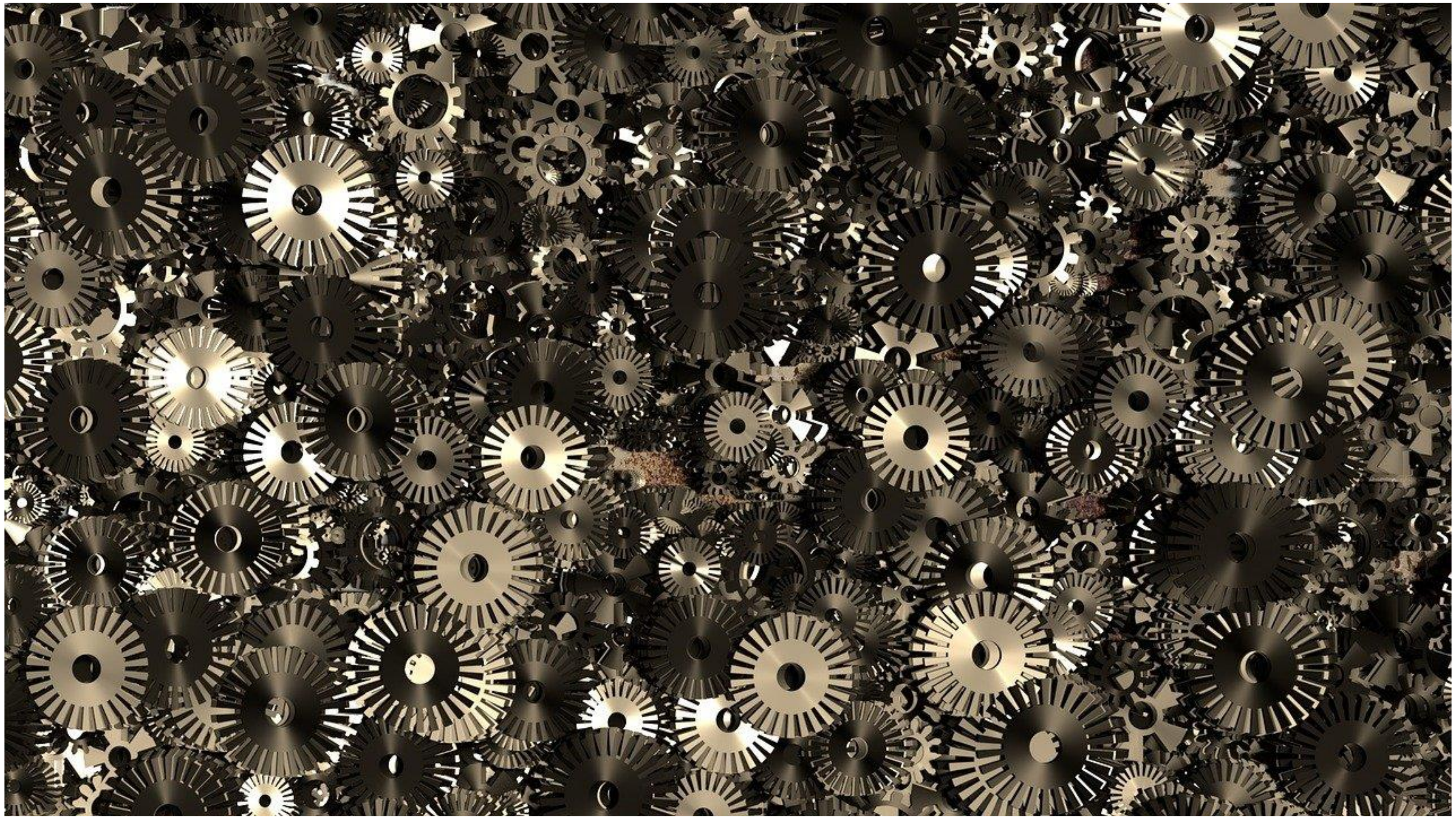

## Outline

1. Wigner's unitary group representations
2. Generators and total observables
3. Irreducible representations and elementary particles
4. Wigner's classification of elementary particles by mass and spin
5. Relativistic analog of one-particle Schrödinger equation

**Sources**:

- S. Weinberg, "The quantum theory of fields", Vol. 1, Chapter 2
- E.S., EPT, Vol. 1, Chapters 3-5

# Symmetries in quantum mechanics

**Wigner theorem:** *Each inertial transformation g is represented by a unitary transformation* $U_g$ *in the Hilbert space*

- If the inertial transformation $g$ is aplied to the state preparation device, then $U_g$ should be applied to the state vector (Schrödinger picture)
$$|\Psi'\rangle = U_g |\Psi\rangle$$
- If the inertial transformation $g$ is applied to the measuring apparatus, then $U_g$ should be applied to the operator of the observable (Heisenberg picture)
$$F' = U_g F U_g^{-1}$$
- If the entire laboratory (state + measuring device) is transformed, then measurement results do not change
$$\langle\Psi'|F'|\Psi'\rangle = \langle\Psi|U_g^{-1}\left(U_g F U_g^{-1}\right)U_g |\Psi\rangle = \langle\Psi|F|\Psi\rangle$$
This is Einstein's principle of relativity

# Symmetries in quantum mechanics


Valentine Bargmann

**Wigner-Bargmann theorem:** *Operators $U_g$ form a unitary representation (single-valued or double-valued) of the Poincaré group in the Hilbert space of the system:*

$$U_{g1}\, U_{g2} = \pm U_{g1g2}$$

Ten generators of the Poincaré Lie algebra are represented by Hermitian operators with clear physical meanings

| Transformation | $U_g$ | Hermitian generator | Interpretation |
|---|---|---|---|
| space translation | $e^{-i\boldsymbol{P}\cdot\mathbf{x}/\hbar}$ | $\boldsymbol{P}$ | total momentum |
| rotation | $e^{-i\boldsymbol{J}\cdot\boldsymbol{\phi}/\hbar}$ | $\boldsymbol{J}$ | total angular momentum |
| time translation | $e^{iH\mathrm{t}/\hbar}$ | $H$ | total energy (Hamiltonian) |
| boost | $e^{-ic\boldsymbol{K}\cdot\boldsymbol{\theta}/\hbar}$ | $\boldsymbol{K}$ | - c.m. position × total energy |

## Example of a group representation

Group: translations $T_a$ in 1 dimension ($a$ = translation distance)

Group multiplication law: $T_a T_b = T_{a+b}$

Hilbert space: 1 dimensional

Unitary operators: unimodular complex numbers

$$U = e^{ix}, x \in \mathbb{R}$$

$$UU^\dagger = e^{ix} e^{-ix} = 1$$

Unitary group representation: $T_a \rightarrow e^{ia/\hbar}$

Verification of the main properties of the group representation:

$$U_a U_b = e^{ia/\hbar} e^{ib/\hbar} = e^{i(a+b)/\hbar} = U_{a+b}$$

$$U_{-a} = e^{-ia/\hbar} = \left(e^{ia/\hbar}\right)^* = U_a^\dagger = U_a^{-1}$$

$$U_0 = 1$$

**Important!** Condition of relativistic invariance:

- A quantum theory is relativistically invariant if and only if its Hilbert space has 10 Hermitian operators $(\boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K}, H)$ satisfying Poincaré commutation relations:

$$[J_i, P_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} P_k, \tag{3.49}$$

$$[J_i, J_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} J_k, \tag{3.50}$$

$$[J_i, K_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} K_k, \tag{3.51}$$

$$[P_i, P_j] = [J_i, H] = [P_i, H] = 0, \tag{3.52}$$

$$[K_i, K_j] = -\frac{i\hbar}{c^2} \sum_{k=1}^{3} \epsilon_{ijk} J_k, \tag{3.53}$$

$$[K_i, P_j] = -\frac{i\hbar}{c^2} H \delta_{ij}, \tag{3.54}$$

$$[K_i, H] = -i\hbar P_i. \tag{3.55}$$

- Building a representation of a group or its Lie algebra are equivalent tasks. The latter is often easier to achieve.

# Other physical observables are expressed as functions of generators

| Observable | Operator |
|---|---|
| mass | $M = \sqrt{H^2 - P^2c^2}/c^2$ |
| c.m. velocity | $\boldsymbol{V} = c^2\boldsymbol{P}/H$ |
| c.m. position (Newton-Wigner) | $\boldsymbol{R} = -(H^{-1}\boldsymbol{K} + \boldsymbol{K}H^{-1})/c^2 + \frac{c[\boldsymbol{P} \times \boldsymbol{W}]}{MH(Mc^2 + H)}$ |
| spin | $\boldsymbol{S} = \boldsymbol{J} - [\boldsymbol{R} \times \boldsymbol{P}]$ |

Where $\boldsymbol{W} = H\boldsymbol{J}/c - c[\boldsymbol{P} \times \boldsymbol{K}]$ is the *Pauli-Lubanski vector*

# Inertial transformations of observables (Heisenberg picture)

With these definitions and commutators you can do a lot of useful physics.
For example:

1. Total momentum is conserved

$$\boldsymbol{P}(t) = e^{iH\mathrm{t}/\hbar}\boldsymbol{P}\, e^{-iH\mathrm{t}/\hbar} = \boldsymbol{P} + \frac{it}{\hbar}[H, \boldsymbol{P}] - \frac{t^2}{2!\hbar^2}\left[H, [H, \boldsymbol{P}]\right] + \cdots = \boldsymbol{P}$$

2. Total  energy are conserved

$$H(t) = e^{iH\mathrm{t}/\hbar}H\, e^{-iH\mathrm{t}/\hbar} = H$$

3. Center of mass of any isolated system moves with a constant velocity

$$\boldsymbol{R}(t) = e^{iH\mathrm{t}/\hbar}\boldsymbol{R}(0)\, e^{-iH\mathrm{t}/\hbar} = \boldsymbol{R}(0) + \boldsymbol{V}t$$

4. Relativistic law of addition of velocities ($v = c\tanh\theta$)

$$V'_x = e^{-icK_x\theta/\hbar}\, V_x\, e^{-icK_x\theta/\hbar} = \frac{V_x - v}{1 - V_x v/c^2}$$

# Irreducible representations and elementary particles

- Reducible representation $U_g = \begin{bmatrix} V_g & 0 \\ 0 & W_g \end{bmatrix} \equiv V_g \oplus W_g$ in some basis

- Irreducible representations = simplest Hilbert spaces describing elementary particles
- Schur's first lemma: In irreducible representation, if $[X, U_g] = 0$ for all $g$, then $X = \alpha I$
- Poincaré Lie algebra has two *Casimir operators* $M^2 = (H^2 - P^2c^2)/c^4$ and $S^2$ which commute with all generators $(\boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K}, H)$, therefore, elementary particles are classified by two parameters $m$ and $s$.

# Irreducible representations and elementary particles

| Particle type | mass | spin $s$, helicity $\tau$ | number of wavefunction components | energy, momentum |
|---|---|---|---|---|
| massive | $m > 0$ | $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ | $2s + 1$ | $H = \sqrt{P^2c^2 + m^2c^4}$ |
| massless (photons) | $m = 0$ | $\tau = \cdots, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$ | 1 | $H = cP$ |
| vacuum | $m = 0$ | $s = 0$ | n/a | $H = 0, \boldsymbol{P} = 0$ |

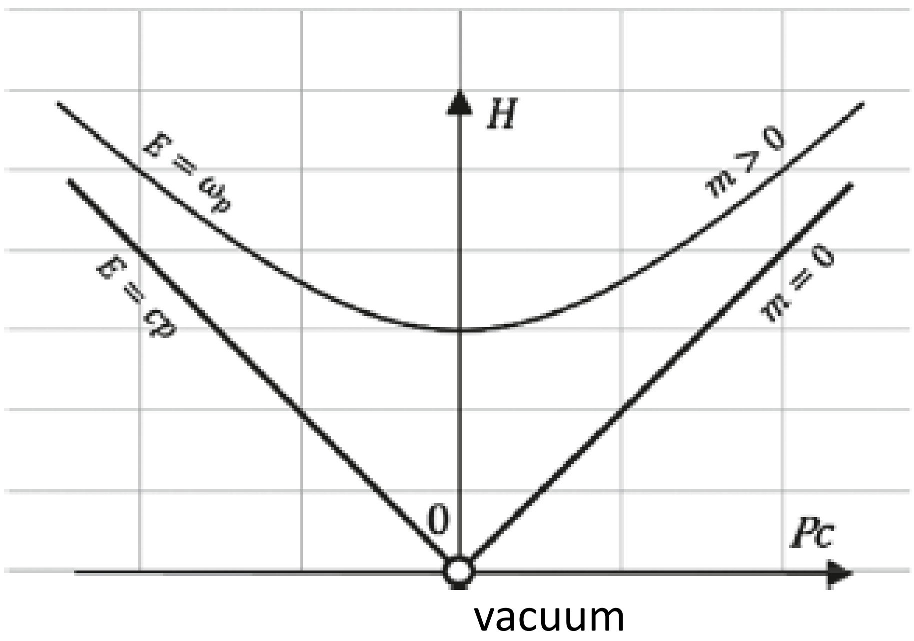


**Figure 5.3:** Mass hyperboloid in the energy momentum space for massive particles and the zero-mass cone for $m = 0$.

## Properties of free elementary particles

1. Speed of a massive particle is less that the speed of light

$$|\boldsymbol{V}| = c^2 P/H = \frac{c^2 P}{\sqrt{P^2 c^2 + m^2 c^4}} < c$$

2. Speed of a massless particle (photon) is equal to the speed of light

$$|\boldsymbol{V}| \ = c^2 P/H = c^2 P/cP = c$$

3. And this is observer-independent (2nd Einstein's postulate)

$$V'_x = \frac{c - v}{1 - cv/c^2} = c$$

# Explicit action of transformations in particle Hilbert space (m>0)

$|\boldsymbol{p}, s_z\rangle$ are eigenvectors of momentum $\boldsymbol{P}$ and spin component $S_z$

$$e^{-\frac{i}{\hbar}\boldsymbol{P}\cdot\boldsymbol{a}}|\boldsymbol{p}, s_z\rangle = e^{-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{a}}|\boldsymbol{p}, s_z\rangle, \quad (5.8)$$

space translations

$$e^{\frac{i}{\hbar}Ht}|\boldsymbol{p}, s_z\rangle = e^{\frac{i}{\hbar}\omega_{\boldsymbol{p}}t}|\boldsymbol{p}, s_z\rangle. \quad (5.9)$$

time translations $\left(\omega_{\boldsymbol{p}} = \sqrt{p^2c^2 + m^2c^4}\right)$

It follows from (5.9) that one-particle wave functions satisfy a relativistic analog of the Schrödinger equation

$$i\hbar \frac{d\psi_{s_z}(\boldsymbol{p}, t)}{dt} = \sqrt{p^2c^2 + m^2c^4}\,\psi_{s_z}(\boldsymbol{p}, t)$$

$$e^{-\frac{i}{\hbar}\boldsymbol{J}\cdot\boldsymbol{\varphi}}|\boldsymbol{p}, s_z\rangle = \sum_{s'_z=-s}^{s} \mathscr{D}^{s}_{s'_z s_z}(\boldsymbol{\varphi})|\boldsymbol{\varphi p}, s'_z\rangle. \quad (5.10)$$

rotations

$$e^{-\frac{ic}{\hbar}\boldsymbol{K}\cdot\boldsymbol{\theta}}|\boldsymbol{p}, s_z\rangle = \frac{N(\boldsymbol{p})}{N(\boldsymbol{\theta p})} \sum_{s'_z=-s}^{s} \mathscr{D}^{s}_{s'_z s_z}(\boldsymbol{\varphi}_W(\boldsymbol{p}, \boldsymbol{\theta}))|\boldsymbol{\theta p}, s'_z\rangle. \quad (5.19)$$

boosts ($\boldsymbol{\varphi}_W(\boldsymbol{p}, \boldsymbol{\theta})$ is Wigner's angle)

where

$$N(\boldsymbol{p}) = \sqrt{\frac{mc^2}{\omega_{\boldsymbol{p}}}}. \quad (5.29)$$

## Conclusions:

- Wigner's theory provides a complete description of
  - single elementary particles
  - total observables of ANY isolated physical system
- Next step: multiparticle systems and interactions there.
- This was accomplished in Dirac's theory of relativistic forms of dynamics (to be discussed in the next lecture).

Thank you!